The reverse Minkovwski integral inequality whith parameters $0 < p < 1$ and $p < 0$.

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Abstract. The aim of this work is to obtain the reverse Minkowski inequality for integral with parameters $0 < p < 1$ and $p < 0$.

0.1 Introduction

In recent years, inequalities are playing a very significant role in all fields of mathematics and present a very active and attractive field of research. As example, let us cite the field of integration which is dominated by inequalities involving functions and their integrals ([3], [4], [5], [6]). One of the famous integral inequalities is Minkowski's integral inequality. In particular the following statement was proved for $p \geq 1$ (for details to see [1]).

Theorem 1. *Let $1 \leq p \leq \infty, \Omega \subset \mathbb{R}^n$ and $A \subset \mathbb{R}^m$ be a measurable sets. Suppose that f is measurable on $\Omega \times A$ and $f(.,y) \in L_p(\Omega)$ for almost all $y \in A$. Then*

$$\| \int_A f(.,y)dy\|_{L_p(\Omega)} \leq \int_A \|f(.,y)\|_{L_p(\Omega)}dy \quad (1)$$

if the right-hand side is finite.

Remark 1. *If $0 < p < 1$, mes $A > 0$ and mes $\Omega > 0$ inequality (1) is not valid (to see [1]).*

In this work we consider the reverse inequality of (1) with $0 < p < 1$ and $p < 0$, for $f : (a,b) \times (c,d) \longrightarrow \mathbb{R}$.

We need the following reverse Hölder's inequality with negative parameter p and $p' = \frac{p}{p-1}$ (p' is the conjugate of p),

$$\int_\Omega |f.g|dx \geq \|f\|_{L_p}\|g\|_{L_{p'}}.$$

(For details to see [2]).

0.2 Main Results

Theorem 2. *Let $0 < p < 1$, $-\infty < a < b < \infty$ and $-\infty < c < d < \infty$. Suppose that f is measurable on $(a,b) \times (c,d)$ and $f(x,y) \in L_p((a,b))$ for almost all $y \in (c,d)$. Then*

$$\left\| \int_c^d f(x,y)dy \right\|_{L_p(a,b)}$$

$$\geq \int_c^d \|f(x,y)\|_{L_p(a,b)}dy \qquad (2)$$

if the left-hand side is finite.

Proof.

$$\left| \int_c^d f(x,y)dy \right| \leq \int_c^d |f(x,y)|\, dy,$$

then

$$\left| \int_c^d f(x,y)dy \right|^{p-1}$$

$$\geq \left(\int_c^d |f(x,y)|\, dy \right)^{p-1} \quad \text{for } p-1 < 0.$$

We have

$$\left| \int_c^d f(x,y)dy \right|^p$$

$$= \left| \int_c^d f(x,y)dy \right|^{p-1} \left| \int_c^d f(x,y)dy \right|$$

$$\geq \left(\int_c^d |f(x,y)|\, dy \right)^{p-1} \left| \int_c^d f(x,y)dy \right|.$$

Therefore

$$\left| \int_c^d f(x,y)dy \right|^p$$

$$\geq \left(\int_c^d |f(x,y)|dy\right)^{p-1} \left|\int_c^d f(x,y)dy\right|. \quad (3)$$

Taking in account the following equality

$$\left|\int_c^d f(x,y)dy\right| = \left|\int_c^d |f(x,y)|dy\right|$$

and by integrating (3), we obtain

$$\int_a^b \left|\int_c^d f(x,y)dy\right|^p dx$$

$$\geq \int_a^b \left(\int_c^d |f(x,y)|\,dy\right)^{p-1} \left|\int_c^d f(x,y)dy\right| dx$$

$$= \int_a^b \left(\int_c^d |f(x,y)|\,dy\right)^{p-1} \left|\int_c^d |f(x,y)|dy\right| dx$$

$$= \int_a^b \left|\int_c^d \left(\int_c^d |f(x,y)|\,dy\right)^{p-1} |f(x,y)|dy\right| dx$$

$$\geq \left|\int_a^b \left\{\int_c^d \left(\int_c^d |f(x,y)|\,dy\right)^{p-1} |f(x,y)|dy\right\} dx\right|$$

$$= \left|\int_c^d \left\{\int_a^b \left(\int_c^d |f(x,y)|\,dy\right)^{p-1} |f(x,y)|dx\right\} dy\right|.$$

Let

$$R_1 = \int_a^b \left(\int_c^d |f(x,y)|\,dy\right)^{p-1} f(x,y)dx.$$

Denote $G(x) = \left(\int_c^d |f(x,y)|\,dy\right)^{p-1}$, therefore

$$\|G(x)\|_{L_{p'}((a,b))}$$

$$= \left(\int_a^b \left|\int_c^d |f(x,y)|\,dy\right|^{p'(p-1)} dx\right)^{\frac{1}{p'}}$$

$$= \left(\int_a^b \left|\int_c^d |f(x,y)|\,dy\right|^{p} dx\right)^{\frac{p-1}{p}}$$

$$= \left\{\left(\int_a^b \left|\int_c^d |f(x,y)|\,dy\right|^{p} dx\right)^{\frac{1}{p}}\right\}^{p-1}$$

$$= \left\|\int_c^d |f(x,y)|\,dy\right\|^{p-1}_{L_p((a,b))}.$$

The last expression is finite (by assumption) then $G(x) \in L_{p'}((a,b))$; now one can apply the reverse Hölder inequality

$$R_1 = \int_a^b \left(\int_c^d |f(x,t)|\,dt\right)^{p-1} |f(x,y)|dx$$

$$\geq \left(\int_a^b \left|\int_c^d |f(x,t)|\,dt\right|^{p'(p-1)} dx\right)^{\frac{1}{p'}} \left(\int_a^b |f(x,y)|^p dx\right)^{\frac{1}{p}}$$

$$= \left(\int_a^b \left|\int_c^d |f(x,t)|\,dt\right|^{p} dx\right)^{\frac{1}{p'}} \left(\int_a^b |f(x,y)|^p dx\right)^{\frac{1}{p}}.$$

Let

$$R_2 =$$

$$= \left(\int_a^b \left|\int_c^d |f(x,t)|\,dt\right|^{p} dx\right)^{\frac{1}{p'}} \left(\int_a^b |f(x,y)|^p dx\right)^{\frac{1}{p}},$$

then

$$\int_c^d R_1 dy \geq \int_c^d R_2 dy$$

since $R_2 > 0$, we get

$$\left|\int_c^d R_1 dy\right|$$

$$\geq \left|\int_c^d R_2 dy\right| = \int_c^d R_2 dy$$

consequently

$$\int_a^b \left|\int_c^d f(x,y)dy\right|^p dx$$

$$\geq \left|\int_c^d R_1 dy\right|$$

$$\geq \int_c^d R_2 dy$$

$$= \int_c^d \left(\int_a^b \left|\int_c^d |f(x,t)|\,dt\right|^{p} dx\right)^{\frac{1}{p'}} \left(\int_a^b |f(x,y)|^p dx\right)^{\frac{1}{p}} dy$$

$$= \left(\int_a^b \left|\int_c^d |f(x,t)|\,dt\right|^{p} dx\right)^{\frac{1}{p'}} \int_c^d \left(\int_a^b |f(x,y)|^p dx\right)^{\frac{1}{p}} dy.$$

It follows that

$$\left(\int_a^b \left|\int_c^d f(x,y)dy\right|^p dx\right) \left(\int_a^b \left|\int_c^d |f(x,t)|\,dt\right|^p dx\right)^{-\frac{1}{p'}}$$

$$\geq \int_c^d \left(\int_a^b |f(x,y)|^p dx\right)^{\frac{1}{p}} dy,$$

we have

$$\left|\int_c^d f(x,y)dy\right|$$

$$= \left|\int_c^d |f(x,y)|\, dy\right|$$

and we conclude that

$$\left[\int_a^b \left|\int_c^d f(x,y)dy\right|^p dx\right]^{\frac{1}{p}}$$

$$\geq \int_c^d \left(\int_a^b |f(x,y)|^p dx\right)^{\frac{1}{p}} dy.$$

Theorem 2 is thus proved.

Theorem 3. *Let $p < 0$, $-\infty < a < b < \infty$ and $-\infty < c < d < \infty$. Suppose that f is measurable on $(a,b) \times (c,d)$ and $f(x,y) \in L_p((a,b))$ for almost all $y \in (c,d)$. Then*

$$\left\|\int_c^d f(x,y)dy\right\|_{L_p(a,b)} \geq \int_c^d \|f(x,y)\|_{L_p(a,b)}\, dy \tag{4}$$

if the left-hand side is finite.

Proof. By using the inequality

$$\left|\int_c^d f(x,y)dy\right| \leq \int_c^d |f(x,y)|\, dy,$$

we have

$$\left|\int_c^d f(x,y)dy\right|^p \geq \left(\int_c^d |f(x,y)|\, dy\right)^p, \quad \text{for } p < 0.$$

By integrating the last inequality, we get

$$\int_a^b \left|\int_c^d f(x,y)dy\right|^p dx$$

$$\geq \int_a^b \left(\int_c^d |f(x,y)|\, dy\right)^p dx$$

$$= \int_a^b \left[\left(\int_c^d |f(x,t)|\, dt\right)^{p-1} \cdot \left(\int_c^d |f(x,y)|\, dy\right)\right] dx$$

$$= \int_a^b \left[\int_c^d \left(\int_c^d |f(x,t)|\, dt\right)^{p-1} |f(x,y)|\, dy\right] dx$$

$$= \int_c^d \left\{\int_a^b \left(\int_c^d |f(x,t)|\, dt\right)^{p-1} |f(x,y)|\, dx\right\} dy. \tag{5}$$

Let

$$R_3 = \int_a^b \left(\int_c^d |f(x,t)|\, dt\right)^{p-1} |f(x,y)|\, dx.$$

By similar arguments to the proof in Theorem 2, we can apply the reverse Hölder inequality

$$R_3 \geq$$

$$\geq \left(\int_a^b \left|\int_c^d |f(x,t)|\, dt\right|^p dx\right)^{\frac{1}{p'}} \left(\int_a^b |f(x,y)|^p dx\right)^{\frac{1}{p}}.$$

Let

$$R_4 = \left(\int_a^b \left|\int_c^d |f(x,t)|\, dt\right|^p dx\right)^{\frac{1}{p'}} \left(\int_a^b |f(x,y)|^p dx\right)^{\frac{1}{p}},$$

therefore

$$\int_a^b \left|\int_c^d f(x,y)dy\right|^p dx$$

$$\geq \int_c^d R_3 dy$$

$$\geq \int_c^d R_4 dy$$

$$= \int_c^d \left(\int_a^b \left|\int_c^d |f(x,t)|\, dt\right|^p dx\right)^{\frac{1}{p'}} \left(\int_a^b |f(x,y)|^p dx\right)^{\frac{1}{p}} dy$$

$$= \left(\int_a^b \left|\int_c^d |f(x,t)|\, dt\right|^p dx\right)^{\frac{1}{p'}} \int_c^d \left(\int_a^b |f(x,y)|^p dx\right)^{\frac{1}{p}} dy.$$

Finally we get

$$\left(\int_a^b \left|\int_c^d f(x,y)dy\right|^p dx\right) \left(\int_a^b \left|\int_c^d |f(x,t)|\, dt\right|^p dx\right)^{-\frac{1}{p'}}$$

$$\geq \int_c^d \left(\int_a^b |f(x,y)|^p dx\right)^{\frac{1}{p}} dy.$$

It follows that

$$\left[\int_a^b \left|\int_c^d f(x,y)dy\right|^p dx\right]^{\frac{1}{p}}$$

$$\geq \int_c^d \left(\int_a^b |f(x,y)|^p dx\right)^{\frac{1}{p}} dy$$

or

$$\left\|\int_c^d f(x,y)dy\right\|_{L_{p,x}(a,b)}$$

$$\geq \int_c^d \|f(x,y)\|_{L_{p,x}(a,b)}\, dy.$$

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