EQUIALENT SEMINORMS ON AN INTERVAL IN THE NIKOLSKII-BESOV SPACES

Halim Benali, Senouci Aek.

University of Tiaret-Algeria.

Abstract:

The aim of this work is to prove the equivalence of the semi- norms $\|.\|_{b^{l}_{p,\theta}(a,b)}$ where (a,b) is an interval.The equivalence is valid for $\sigma = 1,2$.

(for R^n the equivalence is well- known see (3) p.201-208.)

Key words: Inequalities,integrals,Nikolskii-besov spaces

Definition 0.1

Let $l>0$, $\sigma \in N, 1 \le p, \theta \le \infty, -\infty < a < b < \infty$.

The function f belongs to the space $b_{p,\theta}^{l}(a,b)$ if :

1) f is measurable on (a,b) and

2) $\|f\|_{b^{l}_{p,\theta}(a,b)} = \|f\|_{L_p(a,b)} +$

$$\left(\int_0^{\frac{b-a}{\sigma}} \left(\frac{\|\Delta^{\sigma}{}_h f\|_{L_p(a,b)}}{h^l} \right)^{\theta} \frac{dh}{h} \right)^{\frac{1}{\theta}} \prec \infty \qquad (1)$$

$$\|f\|_{b^{l}_{p,\infty}(a,b)} = \sup_{0<h<\frac{b-a}{\sigma}} \frac{\|\Delta^{\sigma}{}_h f\|_{L_p(a,b-\sigma h)}}{h^l} < \infty$$

(2)

Theo rem 0.1

Let $l > 0$,

$\sigma \in N, 1 \le p, \theta \le \infty, -\infty < a < b < \infty$.

The semi norms $\|.\|_{b^{l}_{p,\theta}(a,b)}$ corresponding to different $\sigma \in N$ satisfying $\sigma > l$ are equivalent.

We recall an result well known for R^n.

Définitions 0.2

Let $l>0$, $\sigma \in N, 1 \le p, \theta \le \infty$. The function f belongs to the space $B_{p,\theta}^{l}(R^n)$ if :

1) f is measurable on R^n and

2) $\|f\|_{B_{p,\theta}^{l}(R^n)} = \|f\|_{L_p(R^n)} + \|f\|_{b_{p,\theta}^{l}(R^n)} < \infty$.

Where

$$\|f\|_{b_{p,\theta}^{l}(R^n)} = \left(\int_{R^n} \left(\frac{\|\Delta_h{}^{\sigma} f\|_{L_p(R^n)}}{|h|^l} \right)^{\theta} \frac{dh}{|h|^n} \right)^{\frac{1}{\theta}}$$

$$\|f\|_{b_{p,\infty}^{l}(R^n)} = \sup_{h \neq 0, h \in R^n} \frac{\|\Delta^{\sigma}{}_h f\|_{L_p(R^n)}}{|h|^n} \qquad (4)$$

This definition is independent of $\sigma > l$ as the following lemma shows.

Lemma 0.1

Let $l > 0$, $\sigma \in N, 1 \le p, \theta \le \infty$, The semi-norms

$\| \, \|_{b_{p,\theta}^{l}(R^n)}$ corresponding to different $\sigma \in N$ satisfying $\sigma > l$ are equivalent.

Idea of proof.

Denote temporarily semi-norms $\|.\|_{b_{p,\theta}^{l}(R^n)}$ corresponding to σ by $\|.\|^{(\sigma)}$ .It is enought to prove that $\|.\|^{(\sigma)}$ and $\|.\|^{(\sigma-1)}$ are equivalent on $L_p(R^n)$ where $\sigma > l$ .Since

$$\|\Delta_h{}^{(\sigma+1)} f\|_{L_p(R^n)} \le 2\|\Delta_h{}^{\sigma} f\|_{L_p(R^n)} \quad ....(5)$$

It follows that . $\|.\|^{(\sigma+1)} \le 2\|.\|^{(\sigma)}$. To prove the conerse inequality start whith the case 0<l<1,and apply the following identity for differences

$$\Delta_h f = 2^{-1}\Delta_{2h} f - 2^{-1}\Delta_h^2 f$$

Which is equivalent to the following identity
$(x-1)=2^{-1}(x^2-1)-2^{-1}(x-1)^2$
(here x replaces the translation operator E_h
Where $h\in R^n$):

$E_h f(y)=f(y+h)-f(y), y\in R^n$
To complete the proof deduce a similar identity involving $\Delta_h^{\sigma},\Delta^{\sigma}{}_{2h},\Delta^{\sigma+1}{}_h$.

Proof:

Step1. $\sigma=1$

Supose that 0<l<1 and $\|f\|^2<\infty$. From (5) and by apply Minkovski's inequality we obtain,

$$\|\Delta_h f\|_{L_p(R^n)}\le 2^{-1}\|\Delta_{2h}f\|_{L_p(R^n)}+2^{-1}\|\Delta_h{}^2 f\|_{L_p(R^n)}$$
(6)

first. $\theta=+\infty$

We set $$\|f\|^{(2)}=\sup_{h\in R^n,h\neq 0}\frac{\|\Delta_h{}^2 f\|_{L_p(R^n)}}{|h|^l},$$

And $$\varphi(h)=|h|^{-l}\|\Delta_h f\|_{L_p(R^n)}$$

For all $h\in R^n, h\neq 0;\varphi(h)<\infty$.
From (6) we have,

$$\varphi' h)\le\frac{\|\Delta_{2h}f\|_{L_p(R^n)}}{2|h|^l}+\frac{\|\Delta_{2h}f\|_{L_p(R^n)}}{2|h|^l}\qquad(7)$$

Since $$\varphi(2\mathrm{h})=\frac{\|\Delta_{2h}f\|_{L_p(R^n)}}{2^l|h|^l}$$

It follows that

$$\varphi(\mathrm{h})\le 2^{l-1}\varphi(2h)+2^{-1}\frac{\|\Delta_h{}^2 f\|_{L_p(R^n)}}{|h|^l}$$

$$\varphi(\mathrm{h})\le 2^{l-1}\varphi(2h)+2^{-1}\sup_{h\in R^n,h\neq 0}\frac{\|\Delta_h{}^2 f\|_{L_p(R^n)}}{|h|^l}$$

Then we obtain,
$$\varphi(h)\le 2^{l-1}\varphi(2h)+2^{-1}\|f\|^{(2)}$$
(8)

And
$$\varphi(2h)\le 2^{l-1}\varphi(2^2h)+2^{-1}\|f\|^{(2)}$$
Hence

$$\varphi(h)\le 2^{3(l-1)}\varphi(2^3h)+2^{-1}\|f\|^{(2)}(1+2^{l-1}+2^{2(l-1)})$$

consequently,for all $k\in\mathrm{N}$ we have,
$$\varphi(h)\le 2^{k(l-1)}\varphi(2^kh)+2^{-1}\|f\|^{(2)}(1+2^{l-1}+\dots+2^{k(l-1)})$$
Since l<1 the series $1+\dots+2^{k(l-1)}+\dots$ is convergent and her sum is $2(2-2^l)^{-1}$.
it follows that

$$\varphi(h)\le 2^{k(l-1)}\varphi(2^kh)+(2-2^l)\|f\|^{(2)}\qquad(9)$$
Let k be such that $2^k|h|\ge 1$, then
$$\varphi(2^kh)\le\|\Delta_{2^kh}f\|_{L_p(R^n)}\le 2\|f\|^{(2)}$$
$$\varphi(h)\le 2^{k(l-1)-l}\|f\|_{L_p(R^n)}+(2-2^l)^{-1}\|f\|^{(2)}$$
(10)

On letting $k\to\infty$ in (10) we obtain
$$\varphi(h)\le(2-2^l)^{-1}\|f\|^{(2)}$$
Thus
$$\|f\|^{(1)}\le(2-2^l)^{-1}\|f\|^{(2)}$$
Finally,we get,
$$(2-2^l)\|f\|^{(1)}\le\|f\|^{(2)}\le 2\|f\|^{(1)}$$
(11)

Second $1\prec\theta\prec\infty$.

We set for all $\varepsilon>0$
$$\psi(\varepsilon)=\left[\int_{|h|\ge\varepsilon}\left(\frac{\|\Delta_h f\|_{L_p(R^n)}}{|h|^l}\right)^{\theta}\frac{dh}{|h|^n}\right]^{\frac{1}{\theta}}$$

Clearly $\psi(\varepsilon)<\infty$. Indeed
$$\psi(\varepsilon)\le 2\|f\|_{L_p(R^n)}\int_{|h|\ge\varepsilon}|h|^{-l\vartheta-n}dh<\infty$$

From the inequality (6) and by apply Minkoski's inequality,
we obtain

$$\left[\int_{|h|\geq\varepsilon}\left(\frac{\|\Delta_h f\|_{L_p(R^n)}}{|h|^l}\right)^\theta \frac{dh}{|h|^n}\right]^{\frac{1}{\theta}} \leq$$

$$\frac{1}{2}\left[\int_{|h|\geq\varepsilon}\left(\frac{\|\Delta_{2h} f\|_{L_p(R^n)}}{|h|^l}\right)^\theta \frac{dh}{|h|^n}\right]^{\frac{1}{\theta}}$$

$$+\frac{1}{2}\left[\int_{|h|\geq\varepsilon}\left(\frac{\|\Delta^2{}_h f\|_{L_p(R^n)}}{|h|^l}\right)^\theta \frac{dh}{|h|^n}\right]^{\frac{1}{\theta}}$$

(12)

Hence $\psi(\varepsilon) \leq I_1+I_2$ where

$$I_1=\frac{1}{2}\left[\int_{|h|\geq\varepsilon}\left(\frac{\|\Delta_{2h} f\|_{L_p(R^n)}}{|h|^l}\right)^\theta \frac{dh}{|h|^n}\right]^{\frac{1}{\theta}}$$

$$I_2=\frac{1}{2}\left[\int_{|h|\geq\varepsilon}\left(\frac{\|\Delta^2{}_h f\|_{L_p(R^n)}}{|h|^l}\right)^\theta \frac{dh}{|h|^n}\right]^{\frac{1}{\theta}}$$

We have

$$I_2\leq\frac{1}{2}\left[\int_{R^n}\left(\frac{\|\Delta^2{}_h f\|_{L_p(R^n)}}{|h|^l}\right)^\theta \frac{dh}{|h|^n}\right]^{\frac{1}{\theta}}$$

$$=2^{-1}\|f\|^{(2)}$$

By substituting t=2h in the integral I ,we obtain

$$I_1=\frac{1}{2}\left[\int_{|t|\geq 2\varepsilon}\left(\frac{\|\Delta_t f\|_{L_p(R^n)}}{|t|^l}\right)^\theta \frac{2^{-n}}{2^{-n}}\frac{dt}{|t|^n}\right]^{\frac{1}{\theta}}$$

$$=\frac{1}{2}\left[\int_{|t|\geq 2\varepsilon}\left(\frac{\|\Delta_t f\|_{L_p(R^n)}}{|t|^l}\right)^\theta \frac{2^{-n}}{2^{-n}}\frac{dt}{|t|^n}\right]^{\frac{1}{\theta}}$$

$$=2^{l-1}\psi(2\varepsilon)$$

consequently

$$\psi(\in)\leq 2^{l-1}\psi(2\in)+2^{-1}\|f\|^{(2)}$$

(13)

so we have a an similar inequality to (8) .

Hence,we have for all $k\in N$

$$\psi(\in)\leq 2^{k(l-1)}\psi(2^k\in)+(2-2^l)^{-1}\|f\|^{(2)}$$

On letting $k\to+\infty$ in (14) we obtain

$$\|f\|^{(1)}=\left[\int_{R^n}\left(\frac{\|\Delta_h f\|_{L_p(R^n)}}{|h|^l}\right)^\theta \frac{dh}{|h|^n}\right]^{\frac{1}{\theta}}$$

Then , we get

$$\|f\|^{(1)}\leq(2-2^l)^{-1}\|f\|^{(2)}$$

$$(2-2^l)\|f\|^{(1)}\leq\|f\|^{(2)}\leq 2\|f\|^{(1)}$$

We conclude that for all $0\prec l\prec\sigma=1$ the semi- norms $\|f\|^{(1)},\|f\|^{(2)}$ are equivalent.

Step 2. $\sigma\geq 2$

$$(x-1)^\sigma=2^{-\sigma}(x^2-1)^\sigma+(x-1)^\sigma-2^{-\sigma}(x^2-1)^\sigma$$

$$(x-1)^\sigma=2^{-\sigma}(x^2-1)^\sigma+(x-1)^\sigma-2^{-\sigma}(x^2-1)^\sigma$$

$$(x+1)^\sigma$$

$$(x-1)^\sigma=2^{-\sigma}(x^2-1)^\sigma+2^\sigma 2^{-\sigma}(x-1)^{\sigma+1}(x-1)^{-1}$$

$$-2^{-\sigma}(x-1)^{\sigma+1}(x-1)^{-1}(x+1)^\sigma$$

$$(x-1)^\sigma=2^{-\sigma}(x^2-1)^\sigma-2^{-\sigma}(x-1)^{\sigma+1}(x-1)^{-1}$$

$$((x+1)^\sigma-2^\sigma).$$

We set

$$P_{\sigma-1}(x)=-2^{-\sigma}(x-1)^{-1}((x+1)^\sigma-2^\sigma)$$

then

$$(x-1)^\sigma=2^{-\sigma}(x^2-1)^\sigma+P_{\sigma-1}(x)(x-1)^{\sigma+1}$$

(15)

writing $(x+1)=(x-1)+2$, we have

$$(x+1)^\sigma-2^\sigma=(((x-1)+2)^\sigma-2^\sigma)=$$

$$\sum_{s=1}^{\sigma}\binom{\omega}{s}(x-1)^s 2^{\sigma-s}$$

hence

$$P_{\sigma-1}(x)=-2^{-\sigma}\sum_{s=1}^{\sigma}\binom{\sigma}{s}(x-1)^{s-1}2^{\sigma-s}=$$

$$-\sum_{s=1}^{\sigma}\binom{\sigma}{s}(x-1)^{s-1}2^{-s}$$

(16)

we replace x by the translation operator E_h in (15),we obtain for all

$f\in L_p(R^n)$

$$\Delta^{\sigma}{}_{h} f = 2^{-1}\Delta^{\sigma}{}_{2h} f - P_{\sigma-1}(E_h)\Delta_h^{\sigma+1} f$$

(17)

where $P_{\sigma-1}(E_h) = -\sum_{s=1}^{\sigma}\binom{\sigma}{s} 2^{-s}(E_h - 1)^{s-1}$

by Minkovski's inequality to (17),we obtain

$$\left\|\Delta_h{}^{\sigma} f\right\|_{L_p(R^n)} \leq 2^{-\sigma}\left\|\Delta_{2h}{}^{\sigma} f\right\|_{L_p(R^n)} + \left\|P_{\sigma+1}(E_h)\right\|_{L(L_p(R^n))}\left\|\Delta_h{}^{\sigma+1} f\right\|_{L_p(R^n)}$$

(18)

first . $\theta = +\infty$

we set

$\varphi(h) = |h|^{-l}\left\|\Delta_h^{\sigma} f\right\|_{L_p(R^n)}$ and $c_1 = 2^{-1}(2^{\sigma} - 1)$

For all $h \in R^n, h \neq 0; \varphi(h) < \infty$

from (18) we have

$$\varphi(\mathrm{h}) \leq 2^{l-\sigma}\varphi(2h) + c_1 \frac{\left\|\Delta^{\sigma-1}{}_h f\right\|_{L_p(R^n)}}{|h|^l}$$

(19)

Hence

$$\varphi(h) \leq 2^{l-\sigma}\varphi(2h) + c_1\|f\|^{(\sigma+1)} \qquad (20)$$

consequently for all $k \in N$,we have

$$\varphi(h) \leq 2^{k(l-\sigma)}\varphi(2^k h) + c_1\|f\|^{(\sigma+1)}(1 + 2^{l-\sigma} + ... + 2^{(k-1)(l-\sigma)})$$

And a similar argument leads to the inequality (see the step 1 , $\sigma = 1, \theta = +\infty$)

$$\|f\|^{\sigma} \leq c_1 2^{\sigma}(2^{\sigma} - 2^{l})^{-1}\|f\|^{(\sigma+1)}$$

Consequently

$$c_1^{-1} 2^{-\sigma}(2^{\sigma} - 2^{l})\|f\|^{(\sigma)} \leq \|f\|^{(\sigma+1)} \leq 2\|f\|^{(\sigma)}$$

(21)

Second . $1 \leq \theta \prec +\infty$

We set for all $\varepsilon \succ 0$

$$\psi(\varepsilon) = \left[\int_{|h|\geq\varepsilon}\left(\frac{\left\|\Delta^{\sigma}{}_h f\right\|_{L_p(R^n)}}{|h|^l}\right)^{\theta}\frac{dh}{|h|^n}\right]^{\frac{1}{\theta}}$$

(22)

since

$\left\|\Delta^{\sigma}{}_h f\right\|_{L_p(R^n)} \leq 2^{\sigma}\|f\|_{L_p(R^n)} \prec \infty$ and $l\theta + n \succ n$

we have

$$\psi(\varepsilon) \leq 2^{\sigma}\|f\|_{L_p(R^n)} < \infty$$

from (19)we obtain by apply Minkowski's inequality,we get

$$\left[\int_{|h|\geq\varepsilon}\left(\frac{\left\|\Delta_h f\right\|_{L_p(R^n)}}{|h|^l}\right)^{\theta}\frac{dh}{|h|^n}\right]^{\frac{1}{\theta}} \leq$$

$$\frac{1}{2^{\sigma}}\left[\int_{|h|\geq\varepsilon}\left(\frac{\left\|\Delta_{2h} f\right\|_{L_p(R^n)}}{|h|^l}\right)^{\theta}\frac{dh}{|h|^n}\right]^{\frac{1}{\theta}}$$

$$+ \; c_1\left[\int_{|h|\geq\varepsilon}\left(\frac{\left\|\Delta^2{}_h f\right\|_{L_p(R^n)}}{|h|^l}\right)^{\theta}\frac{dh}{|h|^n}\right]^{\frac{1}{\theta}}$$

(24)

denote $I_1 and I_2$ respectively the first and the second integral of second member of the inequality (24)

we have

$$I_2 \leq c_1\|f\|^{(\sigma+1)}$$

By changing variables t=2h,we get

$$I_2 = 2^{l-\sigma}\psi(2\varepsilon)$$

It follows that

$$\psi(\varepsilon) \leq 2^{l-\sigma}\psi(2\varepsilon) + c_1\|f\|^{(\sigma-1)}$$

(25)

consequently for all $k \in N$

$$\psi(\in) \leq 2^{k(l-\sigma)}\psi(2^k\varepsilon) + c_1\|f\|^{(\sigma+1)} 2^{\sigma}(2^{\sigma} - 2^{l})^{-1}$$

we set $c_2 = c_1 2^{\sigma}(2^{\sigma} - 2^{l})^{-1}$; for all

$\varepsilon \succ 0, \psi(2^k\varepsilon) \prec +\infty$

on letting $k \to +\infty$ in the inequality

$$\psi(\varepsilon) \leq 2^{k(l-\sigma)}\psi({}^k\varepsilon) + c_2\|f\|^{(\sigma+1)}$$

then we obtain for all $\varepsilon \succ 0, \psi(\varepsilon) \leq c_2\|f\|^{(\sigma+1)}$

and ε is arbitrary,then we get

$$\|f\|^{(\sigma)} \leq c_2\|f\|^{(\sigma+1)}$$

$$c_2^{-1}\|f\|^{(\sigma)} \leq \|f\|^{(\sigma+1)} \leq 2\|f\|^{(\sigma)}$$

(26)

So the semi norms are equivalent for all $,\sigma \in N, \sigma \succ l.$

Definition 03.

Let Ω be a bounded open sub set of R^n. The set $\Omega_{\sigma|h|}$ is defined by

$$\Omega_{\sigma|h|} = \{x \in \Omega : dis(x;\partial\Omega) > \sigma|h|\}$$

Definition 04.

As usual for any function $f \in L_p(R^n)$, and $,\sigma \in N$. we denote by $\Delta^{\sigma}{}_{h,\Omega} f$ the th forward difference relative to Ω which is defined by

$$\Delta^{\sigma}{}_{h,\Omega} f(x) = \Delta^{\sigma}{}_h f(x) \quad \text{If } x \in \Omega_{\sigma|h|},$$

$$\Delta^{\sigma}{}_{h,\Omega} f(x) = 0 \quad \text{If } x \notin \Omega_{\sigma|h|}, \qquad (27)$$

where the difference $\Delta^{\sigma}{}_h f$ is the usual difference defined on R^n:

$$\Delta_h{}^{\sigma} f(x) = \sum_{s=1}^{\sigma} \binom{\sigma}{s} (-1)^{\sigma-s} f(x+sh)$$

Proposition 01.

Let Ω be an bounded open sub set of R^n and $f \in L_p(R^n)$, $,\sigma \in N$. Then

$$\left\|\Delta_h{}^{\sigma} f\right\|_{L_p(\Omega_{\sigma|h|})} \le 2^{\sigma} \|f\|_{L_p(\Omega)} \qquad (28)$$

*In particular for an interval (a,b):

$$\left\|\Delta_h{}^{\sigma} f\right\|_{L_p((a,b)_{\sigma|h|})} \le 2^{\sigma} \|f\|_{L_p(a,b)}$$

(29)

Proof of theorem 01.

Denote temporarily the semi norms $\|\ \|_{b^l p,\theta(a,b)}$ corresponding to σ by $\|.\|^{(\sigma)}$

It is enough to prove that $\|.\|^{(\sigma)}$ and $\|.\|^{(\sigma+1)}$ are equivalent on $L_p(a,b)$ where $,\sigma \in N, \sigma \succ l.$

Since

$$\left\|\Delta^{\sigma+1}{}_h f\right\|_{L_p(a,b-(\sigma+1)h)} \le 2 \left\|\Delta^{\sigma}{}_h f\right\|_{L_p(a,b-\sigma\ )} \prec \infty$$

It follows that $\|f\|^{(\sigma+1)} \le 2\|f\|^{(\sigma)}$

proof of the converse inequality ,let $x \in' a,b)$ and denote by

$\widetilde{\Delta}_h{}^{\sigma} f = \Delta^{\sigma}{}_{h,(a,b)} f$ then

$$\widetilde{\Delta}_h f'(x) = f(x+h) - f(x) \quad \text{if } x \in (a,b-h)$$

$$(30) \qquad \widetilde{\Delta}_h f(x) = 0 \quad \text{if } x \notin (a,b-h)$$

and

$$\widetilde{\Delta}_{2h} f'(x) = f(x+2h) - f(x) \quad \text{if } x \in (a,b-2h)$$

$$(31) \qquad \widetilde{\Delta}_{2h} f(x) = 0 \quad \text{if } x \notin (a,b-2h)$$

$$\widetilde{\Delta}^2{}_h f'(x) = f(x+2h) - 2f(x+h) + f(x) \quad \text{if } x \in (a,b-2h)$$

$$(32) \qquad \widetilde{\Delta}^2{}_h f(x) = 0 \quad \text{if } x \notin (a,b-2h)$$

$x \in (a,b-2h)$, then we obtain

$\widetilde{\Delta}_h f'(x) = f(x+h) - f(x)$ and

$2^{-1}\widetilde{\Delta}_{2h} f(x) - 2^{-1}\widetilde{\Delta}_h{}^2 f(x) = 0$

Thus the method considered for R^n is not valid for an interval.Then.we introduce the extension operator for functional space under certain assumptions on an open sub set Ω of R^n (for example extensions for Lipschitz boundaries).

The operator

(33)

$$T : B_{p,\theta}{}^l(a,b) \to B_{p,\theta}{}^l(R)$$

is said extension operator if

$$(34) \qquad Tf/(a,b)(x) = f(x) \quad \text{for all } f \in B_{p,\theta}{}^l(a,b)$$

The extension theorem ensures the existence of the extension operator T(bounded linear) such that for all $p, \theta \ge 1$,

(35)

$$\|Tf\|_{B_{p,\theta}{}^l(R)} \le c\|f\|_{B_{p,\theta}{}^l(a,b)}$$

where c is an positive constant independent of the function .

Then

(36)

$$\|Tf\|^{(\sigma-1)}_{B_{p,\theta}^{l}(R)} \le c\|f\|^{(\sigma+1)}_{B_{p,\theta}^{l}(a,b)}$$

by lemma 01 (case n=1) we obtain)

$$\|Tf\|^{(\sigma)}_{(R)} \le c'\|Tf\|^{(\sigma-1)}_{(R)}$$

by (33),(34) and the inequality

$$\|Tf\|^{(\sigma)}_{(a,b)} \le \|Tf\|^{(\sigma)}_{(R)}$$

we obtain

$$\|Tf\|_{(a,b)}^{(\sigma)} \le \|Tf\|_{R}^{(\sigma)} \le c'\|Tf\|_{(R)}^{(\sigma+1)} \le cc'\|f\|_{(a,b)}^{(\sigma+1)}$$

Hence

$$\|Tf\|^{(\sigma)}_{(a,b)} \le k\|f\|^{(\sigma)}_{(a,b)}$$

where k=cc',

thus

$$\|f\|^{(\sigma)}_{(a,b)} \le k\|f\|^{(\sigma+1)}_{(a,b)}$$

Conclusion: The new fact in this work we introduce the extension operator for functional spaces then we obtain a new result for $\sigma = 1,2$.

References

(1) Besov.O.V.,Il'in,V.P.,Nikol'skii.S.M.Integral representations of function and embedding theorems.Nauka.Moscow,1975(inrussian).Sec,ed,Nauka,Moscow,1996(inrussian),(English.Trans of 1st .Edition,vol1,2 Wiley, Chichester,1979.

(2) Besov.O.V.,sur l'extension des functions avec conservation de certaines proprietés.journal de maths T.58(100)675-684(1962).

(3) Burencov.V.I.Sobolev spaces on domains,Teubner-Texte zur Mathematik,Band 77,Stuttgart-Lepzig,1998.

(4) V?K.Dzidiak.Sur l'extension des fonctions verifiant la condition de Lipschitz dans la metrique , journal de maths 40(_1956),239-242;