The Value of History in a Mathematics Classroom

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Abstract — The history of mathematics, used well, can be a powerful tool in the mathematics classroom. The historical process of mathematical discovery and verification shares many similarities with the process of student learning. We discuss the critical aspects of the learning experience that may be aided through the use of carefully chosen historical parallels, provide historical examples for these aspects, and consider some of the potential pitfalls.

Keywords—history; pedagogy; mathematics

I. INTRODUCTION

We've all experienced the problem. Given the mass of material we are required to cover in our math classes, it seems all but impossible to find avenues for creativity in our lectures. When small time windows open up, we tend to show extra problems, or new applications, or some favourite theoretical wrinkle that we had been saving for such an occasion. Why bring in history? It takes time and effort, and displaces other subjects. What's the advantage?

Simply put, history provides a path for the *entire* mathematical experience. Typically, our students are asked to solve problems and prove theorems, a limited part of what mathematicians do. The full story involves ***motivation:*** what is the context within which the subject arose, and why is it so appealing that it deserves our attention? Next is ***research:*** once the problem is identified, how do we articulate lines of attack that have already been made that might be adapted to the new situation? Third is ***critical thinking:*** how do we transition from received knowledge to new situations? Finally, we have ***implications:*** how does the solution affect us, the academic community, or society? Good history of mathematics synthesizes all these aspects. Bringing it into the classroom can provide for our students a much broader and deeper mathematical experience. Most crucially, history is a *natural* means to attain these goals: we follow real people, who struggled as our students do, and eventually (usually) triumphed. We learn best through stories, and true stories are often the best ones.

In the following, we provide examples of historical episodes to support each of these four aspects of mathematical development.

II. MOTIVATION

All mathematical subjects arose due to some need, either from within mathematics or from outside of it. These needs provide a reason to approach the subject, and to approach it in a certain way.

Example: Trigonometry was invented in ancient Greece to convert geometric models of the motions of the planets into quantitative predictions. The first trigonometric problem, likely solved by Hipparchus of Rhodes, was to determine the

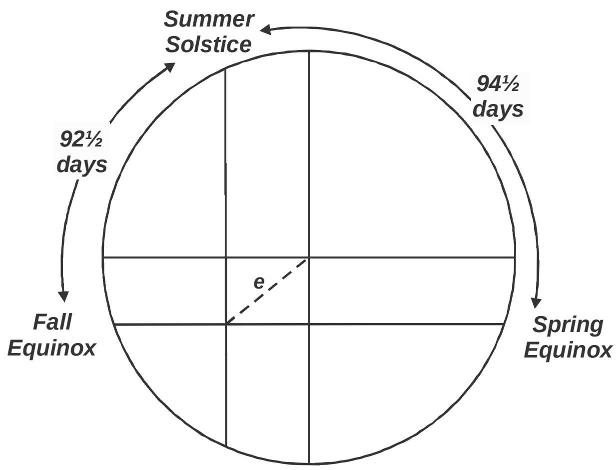


Fig. 1. Hipparchus's model of the motion of the Sun.

eccentricity of the Sun's orbit around the Earth. In Figure 1, the Sun revolves around the orbit circle at a uniform speed; yet Hipparchus observed that the spring and summer are longer than the fall and winter. The goal is to place the Earth a certain distance and direction from the center of the orbit so that the lengths of spring and summer predicted by the model match their observed lengths. This distance e is the eccentricity of the Sun's orbit. Hipparchus and his successors were aided in this calculation by a table, not of sines, but of chord lengths in a circle (Figure 2). The sine would be invented later, in India. Today, trigonometry is still a significant tool for moving back and forth between geometry and numerical measurement.[1]

[1] The earliest record, both of Hipparchus's solar model (pp. 153-155) and of the calculation of a table of chords (pp. 48-60), may be found in Ptolemy's *Almagest* [17]. A popular

Example: The need for a mathematical subject may not have been the same in the past as it is today. Logarithms, for example, were invented in the early 17th century by John Napier as a calculation device for astronomers to reduce the work

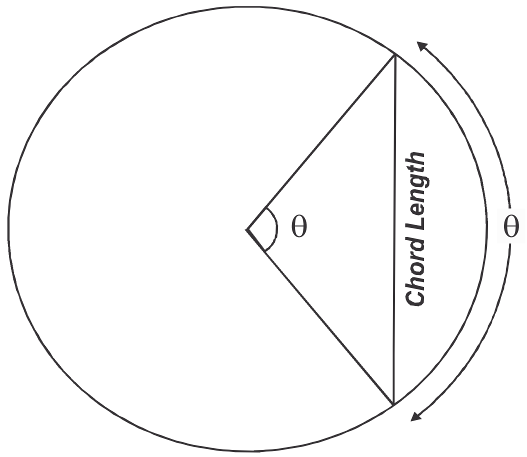


Fig. 2. The definition of a chord, the trigonometric function in ancient Greek astronomy.

involved in finding products and roots of numbers. (The cover of the book announcing Napier's achievement, *Mirifici logarithmorum canonis descriptio*, is shown in Figure 3.) Formulas in spherical astronomy, such as

$$\sin \delta = \sin \lambda \sin \epsilon$$

(which finds the Sun's declination δ from its longitude λ and the obliquity of the ecliptic ε), required astronomers to multiply or divide values of trigonometric functions, usually given to many decimal places, to solve for an unknown quantity. The use of the formula

$$\log(a \cdot b) = \log a + \log b,$$

accompanied by a table of values of logarithms, allows the astronomer to convert the problem of multiplication into the much simpler task of addition. This was such a boon that, almost two centuries later, Pierre Simeon de Laplace said, "by shortening the labours [logarithms] doubled the life of the astronomer." Today our computing power renders this use obsolete. Nevertheless, this historical route can provide a meaningful context for students' first exposure to the subject; the benefits of the theory are obvious, even if its original motivation is no longer active.[2]

account of the chord table is in [1], pp. 121-126.

[2] Napier's *Descriptio* [12] is available on Google Books. See [9] for a treatment of Napier's logarithms adapted to the classroom (although not including Napier's trigonometry).

Fig. 3. The title page of John Napier's 1614 announcement of his invention of logarithms, Mirifici logarithmorum canonis description.

III. Research

Coming to terms with methods that have been devised to

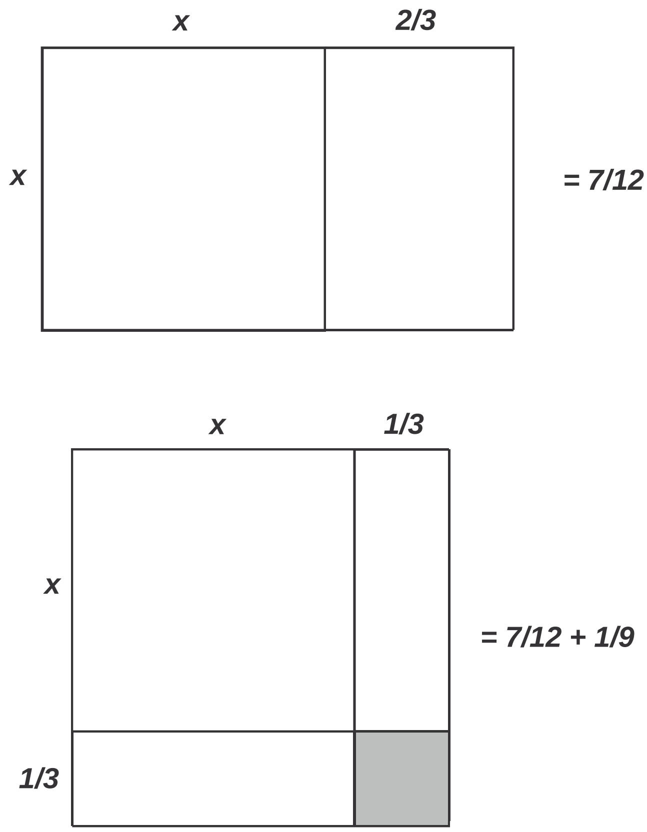


Fig. 4. A cut-and-paste method for solving a quadratic equation.

attack difficult problems is, by definition, a study in history. Techniques helpful in a historical situation may be applied elsewhere, and observation of successful processes of discovery can change a student's perspective on their own searches for solutions.

Example: Ancient Babylonian school children were already solving problems akin to quadratic equations. Although we do not have records of the original discovery of their methods, it seems clear that they must have been based on a “cut-and-paste” geometry. For instance, one such tablet solves the problem

$$x^2 + \frac{2}{3}x = \frac{7}{12}$$

using a computational prescription that mimics precisely the following argument (Figure 4): the left side of the equation can be represented as a square of unknown size appended to a rectangle measuring 2/3 by *x*; the combined figure is asserted to have an area of 7/12. Half the rectangle is cut off and moved to the bottom of the figure. The entire figure is filled in --- we “complete the square” --- by adding the small square in the bottom right. Then the area of the entire square is $\frac{7}{12} + \left(\frac{1}{3}\right)^2 = \frac{25}{36}$, so the side length is 5/6. Subtracting 1/3 gives $x = \frac{1}{2}$. Here we see that converting a numerical problem may gain new significance and an avenue to a solution when it is converted to a geometric arena.[3]

Example: Early in the history of calculus, it was realized that certain differential equations could be easily solved via the discovery that there exists a function that is its own derivative ($f(x) = e^x$). For instance, the standard differential equation for population growth,

$$\frac{dP}{dt} = kP,$$

may be seen to have solution $P = ce^{kt}$ by substituting into the differential equation. This method (taking a variant of the standard exponential function, inserting it into the equation, and altering the parameters to satisfy the equation) was exploited repeatedly to solve many problems from the 18th century onward. As mathematicians such as DeMoivre, Cotes, and Euler started to allow complex quantities to enter the equations through relations such as

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

differential equations representing periodic phenomena became accessible to the same method.[4]

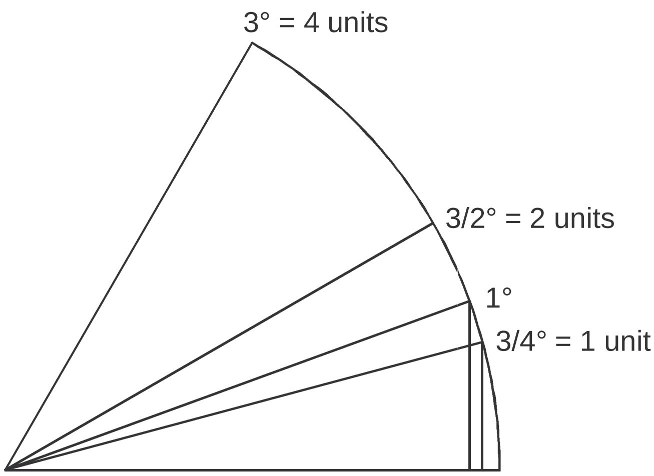


Fig. 5. Sines of small arcs. Most astronomers used degrees, which led to the unfortunate result that the sine of 1° (the larger of the two vertical lines) could not be computed precisely. Al-Samaw’al divided the circle into 480 units, so that the sine of 1 unit (equal to 3/4°) was attainable using geometric methods.

IV. Critical Thinking

Every mathematical community makes shared decisions about the validity and power of various competing approaches. For instance, medieval Indian mathematics valued solutions that we might describe as approximate or iterative, while ancient Greek and medieval Islamic mathematicians preferred direct arguments and calculations. One of the most difficult concepts for modern students to understand is that such commitments are also present today. In order to think creatively, one needs to make informed judgments about alternate avenues of attack; one must know what the community's rules are before one decides to bend or break them.

Example: The 12th-century Iranian scholar Ibn Yaḥyā al-Samaw'al al-Maghribī, late in his life, composed a book entitled *Exposure of the Errors of the Astronomers*, in which he pointed out dozens of places in the works of his colleagues and ancestors where he perceived violations of correct mathematical practice, or downright mistakes. One of these episodes, the calculation of a table of sines, requires finding a value for $\sin 1^\circ$ from a known value of $\sin 3^\circ$. To find a precise solution turns out to require the solution of a cubic equation, and this implies that the problem cannot be solved with geometric methods --- it is equivalent to trisecting an angle. But trigonometry is applied geometry. Since Claudius Ptolemy's time, astronomers had been forced into approximate methods, violating the geometric spirit of the subject. Al-Samaw'al's creative solution was to redefine the number of degrees in a circle from 360 to 480. Then $\sin 3^\circ$ becomes the sine of 4 of the new units (see Figure 5), and applying the sine half angle identity to it twice yields the sine of 1 unit --- thereby bypassing the necessity of approximation altogether.[5]

[3] A translation of this tablet and description of the student’s method appear in [1], pp. 23-24. See [16] for an updated treatment of Mesopotamian mathematics in general.

[4] For a survey of Euler’s mathematics, including much related to differential equations, see [5]. [6], pp. 268-281, focuses on Euler’s contributions to the calculus. See [3] for historical reflections on the role of applications in the development of differential equations in the 18th century.

[5] [18] is an account of al-Samaw’al’s novel approach; it also includes a demonstration that Samaw’al actually did not compute his sine table as he claims.

Fig. 6. Title page of the first edition of Bartholomew Pitiscus's *Trigonometriae* (1600).

Example: The cubic equation was solved in full generality by Gerolamo Cardano in 1545. However, in the case of certain equations such as

$$x^3 = 15x + 4,$$

his method produced nonsensical solutions like

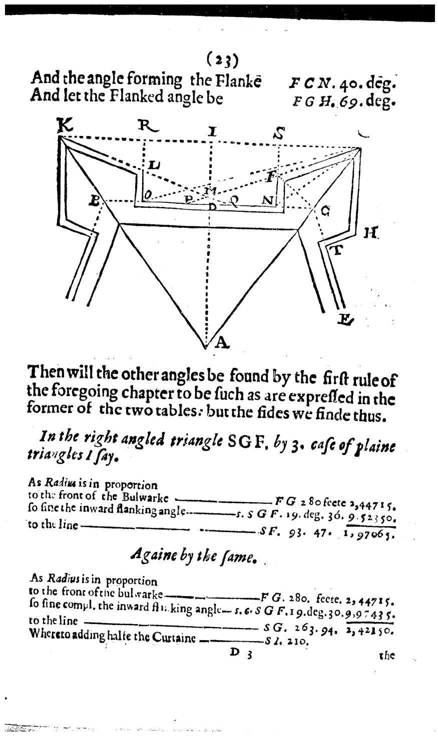
$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}.$$

Rafael Bombelli, an accomplished algebraist, chose to break with Cardano and admit the possibility of the existence of square roots of negative numbers. He proceeded to develop the fundamental laws of complex numbers, and was able to demonstrate that the solution above reduces to $x = 4$.[6]

V. IMPLICATIONS

It is often said that the most powerful mathematical results are those that lead to new and interesting questions or that open mathematics to new applications. Witnessing the enlargement of the social role of certain types of mathematics can be a meaningful lesson in measuring its cultural significance.

Example: In early modern Europe the unification of trigonometry with logarithms in 1614 was motivated by the computational struggles of astronomers such as Tycho Brahe and Johannes Kepler. However, the new tool quickly found applications in a variety of disciplines. In the three decades before 1614, trigonometry had already been applied haltingly to altimetry (finding the heights of buildings and landmarks), surveying, and navigation. The title page of Bartholomew Pitiscus's 1600 *Trigonometriae* (see Figure 6), incidentally the title that coined the word, prominently refers to uses in geodesy,


(23)

And the angle forming the Flankē — F C N. 40. deg.
And let the Flanked angle be — F G H. 69. deg.

Then will the other angles be found by the first rule of the foregoing chapter to be ſuch as are expreſſed in the former of the two tables: but the ſides we finde thus.

In the right angled triangle SGF. *by 3. caſe of plaine triangles I ſay.*

As *Radius* is in proportion
to the front of the Bulwarke ———— *F G* 280 feete 2,44715.
ſo ſine the inward flanking angle ———— *s. S G F.* 19. deg. 36. 9.52350.
to the line ———— *S F.* 93. 47. 1,97065.

Againe by the ſame.

As *Radius* is in proportion
to the front of the bulwarke ———— *F G.* 280. feete. 2,44715.
ſo ſine compl. the inward flanking angle — *s. c. S G F.* 19. deg. 30. 9,97435.
to the line ———— *S G.* 263. 94. 2,42150.
Whereto adding halfe the Curtaine ———— *S I.* 210.

D 3 the

Fig. 7. A page from Richard Norwood's *Fortification, or Architecture Military* (1639).

altimetry, geography, the theory of sundials, astronomy, and (in a later edition) architecture. This was a revolutionary contact of mathematics with the physical world. The introduction of logarithms accelerated the acceptance of mathematics into practical domains. Figure 7, for instance, shows an image from Richard Norwood's 1639 *Fortification, or Architecture Military* demonstrating how to construct military bases using the new computational science. In a very short time, mathematics had moved from the Platonic realm and from the heavens down to the earth. Through this episode, mathematics was transformed into an engine that eventually helped to reshape modern culture through science and technology.[7]

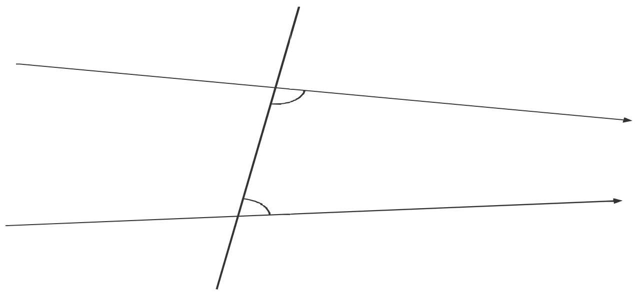


Fig. 8. Euclid's fifth postulate asserts:

[6] [11] is an accessible history of the birth and rise of complex numbers. For a classroom unit on Cardano and Bombelli, see [4].

[7] Pitiscus's *Trigonometriae* [15] may be found on Google Books; Norwood's *Fortification* [13] is in Early English Books Online.

Euclid's fifth postulate asserts: (fig 8) if the two indicated angles are less than two right angles, then the two lines with arrows, if extended, will meet on the side of the arrows.

Example: From the beginning of geometry until the past two centuries, it was assumed (both implicitly and explicitly) that space is Euclidean --- that is, that Euclid's parallel postulate holds. This asserts that, in Figure 8, if the two indicated angles add to less than two right angles, then the two lines meet in the direction indicated by the arrows. This turns out to be logically equivalent to the statement that the sum of angles in any triangle is 180º. After dozens of failed attempts to prove these claims, three separate mathematicians in the early 19th century (Carl Friedrich Gauss, Nicolai Lobachevsky, and János Bolyai) began to consider the possibility that the parallel postulate may be false --- or, at least, that a new and consistent geometry may be born of the assumption of the postulate's contradiction. Through this bold step, they created "worlds out of nothing", elliptical and hyperbolic geometry. These non-Euclidean geometries remained creations of the mind for decades. However, starting in the early 20th century they have become candidate models for the geometry of the universe in which we live. Also, significantly, this episode has revealed that seemingly obvious assumptions that survive for millennia (whether in mathematics or elsewhere) are not necessarily on solid ground.[8]

Students aware of cosmic shifts such as these participate in a much richer educational experience. They are better able to orient themselves and their chosen discipline in a significant place in the intellectual landscape, and are able to act in their profession with more reflectiveness and purpose.

There is one additional aspect of mathematical work that history can support: ***communication***. Since history encompasses entire narratives from initial conception to final product and societal impact, there is a unique opportunity here to improve students' ability to write and otherwise present ideas. Students can write essays; they can make presentations on the background and significance of sub-disciplines; they can write short-answer responses to questions about the significance of and interconnections between theories. It is usually difficult to find opportunities to improve mathematics students' rhetorical skills; history provides a powerful solution.

VI. Challenges

Although the potential benefits of history are diverse, several dangers must be avoided.

a. Misunderstanding history as mere biography

Textbooks often give snapshots of mathematicians' lives and works in the page margins, mistakenly believing they have done a service to history. They have not. Many of these biographies are unrelated in any direct way to the narrative in the text, and so they unintentionally reinforce the tacit misconception that the mathematics itself is ahistorical. Genuine history in the classroom should be *part* of the presentation of the mathematics; its benefits can only be realized with deeper integration.

b. Entering history without sufficient depth

The history of mathematics is a deeply challenging endeavour, requiring sophistication in two disciplines with very different aims and modes of thought. Unfortunately, not everything one finds in the library or online is reliable, either historically or mathematically. The mathematics teacher should consult reliable sources; looking up reviews in professional journals is an effective way to screen out low-quality content.

c. Assuming that history is a universal panacea

Although history is helpful in learning many mathematical concepts, assuming that it *always* leads to positive results is dangerous. Choose moments where the historical context genuinely interacts with the subject, and is appropriate to students' concerns and maturity levels.

VII. Places to Start

For topics in the undergraduate curriculum, there is no better place to begin than Victor Katz's history of mathematics textbooks [10]. For accuracy, mathematical rigour, and thorough coverage, they are unsurpassed; and they provide many connections to the rest of the literature. Also consider a new book to be released in 2016 at MAA Press, based on the history of mathematics course at the Open University in the UK. At an elementary level, consider William Berlinghoff and Fernando Gouvea's *Math Through the Ages* (2nd edition) [2]. Finally, the *MAA Notes* series has published a number of volumes of historical episodes ready for classroom use, edited by Victor Katz, Amy Shell-Gellasch, Dick Jardine, and others (see for example [4], [8], and [9]).

Many modern theories of education attempt to address the plague of passivity in our students by promoting active educational experiences, such as the Moore method and inquiry-based learning. History provides the kind of engagement these innovations attempt to foster. However, history can also enhance the traditional mathematics classroom. By considering the entire cycle of mathematical development, and by asking students not merely to perform calculations but also to reflect upon them, history makes students more powerful, more thoughtful, and more significant. In short, it makes them better mathematicians.

Acknowledgments

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[8] [7] is a textbook on the rise of non-Euclidean geometries aimed at the upper undergraduate level. [8] provides a summary of non-Euclidean geometry, its impacts on physics, philosophy, and art, and its role in the classroom.

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