An Antiplane Electro-Elastic Problem with the Power-Law Friction

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Abstract— *In this paper the material used is electro-elastic and the friction and it is modeled with Tresca's law and the foundation is assumed to be electrically conductive. First we derive the well posedness mathematical model. In the second step, we give the classical variational formulation of the model which is given by a system coupling an evolutionary variational equality for the displacement field and a time-dependent variational equation for the potential field. Then we prove the existence of a unique weak solution to the model by using the Banach fixed-point Theorem.*

Keywords— *Tresca's friction, electro-elastic material, variational inequality, weak solution, fixed point, antiplane shears deformation.*

Mathematics Subject Classification— ***74G25, 49J40, 74F15, 74M10***

1. INTRODUCTION

We consider the antiplane contact problem for electro-elastic materials with Tresca friction law. In this new work, we assume that the dispalcement is parallel to the generators of the cylinder and is dependent of the axial coordinate. Our interest is to describe a physical process (for more details see [1, 4, 5, 6, 7, 8]) in which both antiplane shear, contact, state of material with Trescafriction law and piezoelectric effect are involved, leading to a well posedness mathematical problem. In the variational formulation, this kind of problem leads to an integro-differential inequality. The main result we provide concerns the existence of a unique weak solution to the model, see for instance [2, 3, 6] for details.

The rest of the paper is structured as follows. In Section s:2 we describe the well posedness mathematical model of the frictional contact process between electro-elastic body and a conductive deformable foundation. In Section s:3 we derive the variational formulation. It consists of a variational inequality for the displacement field coupled with a time-dependent variational equation for the electric potential. We state our main result, the existence of a unique weak solution to the model in Theorem 3.1. The Proof of the Theorem is provided in the end of Section s: 4, where it is based on arguments of evolutionary inequalities, and a fixed point Theorem.

2. THE MODEL

In this section, we consider a piezoelectric body $\mathcal{B}$ identified with a region in $\mathbb{R}^3$ it occupies in a fixed and undistorted reference configuration. We assume that $\mathcal{B}$ is a cylinder with generators parallel to the x_3 -axes with a cross-section which is a regular region Ω in the x_1 , x_2 -plane, $Ox_1x_2x_3$ being a Cartesian coordinate system. The cylinder is assumed to be sufficiently long so that the end effects in the axial direction are negligible. Thus, $\mathcal{B}=\Omega\times(-\infty,+\infty)$. The cylinder is acted upon by body forces of density $\mathbf{f}_0$ and has volume free electric charges of density q_0 . It is also constrained mechanically and electrically on the boundary. To describe the boundary conditions, we denote by $\partial\Omega=\Gamma$ the boundary of Ω and we assume a partition of Γ into three open disjoint parts Γ_1 , Γ_2 and Γ_3 , on the one hand, and a partition of $\Gamma_1\cup\Gamma_2$ into two open parts Γ_a and Γ_b , on the other hand. We assume that the one-dimensional measure of Γ_1 and Γ_a , denoted meas Γ_1 and meas Γ_a , are positive.

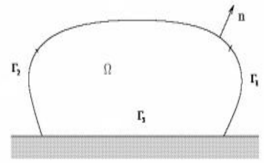

FIGURE 1. Deformable solid Ω on contact with a rigid foundation

The cylinder is clamped on $\Gamma_1\times(-\infty,+\infty)$ and therefore the displacement field vanishes there.

Surface tractions of density $\mathbf{f}_2$ act on $\Gamma_2\times(-\infty,+\infty)$. We also assume that the electrical potential vanishes on $\Gamma_a\times(-\infty,+\infty)$ and a surface electrical charge of density q_2 is prescribed on $\Gamma_b\times(-\infty,+\infty)$. The cylinder is in contact over $\Gamma_3\times(-\infty,+\infty)$ with a conductive obstacle, the so called foundation. The contact is frictional and is modeled with Tresca's law. We are interested in the deformation of the cylinder on the time interval $[0,T]$. We assume that

$$\mathbf{f}_0=(0,0,\ f_0)\ with\ f_0=f_0(x_1,\ x_2,t):\ \Omega\times[0,T]\to \mathrm{IR},\quad (1)$$

$$\mathbf{f}_2=(0,0,\ f_2)\ with\ f_2=f_2(x_1,\ x_2,t):\ \Gamma\times[0,T]\ \to \mathrm{IR},\quad (2)$$

$$q_0=q_0(x_1,\ x_2,t)\ :\ \Omega\times[0,T]\to \mathrm{IR},\quad (3)$$

$$q_2=q_2(x_1,\ x_2,t):\ \Gamma_b\times[0,T]\to \mathrm{IR}.\quad (4)$$

The forces (1), (2) and the electric charges (3), (4) would be expected to give rise to deformations and to electric charges of the piezoelectric cylinder corresponding to a displacement $\mathbf{u}$ and to an electric potential field φ which are independent on x_3 and have the the form

$$\mathbf{u}=(0,0,\ u)\qquad with\qquad u=u(x_1,\ x_2,t):\ \Omega\times[0,T]\to \mathrm{IR},\quad (5)$$

$$\varphi=\varphi(x_1,\ x_2,t)\ :\ \Omega\times[0,T]\to \mathrm{IR}.\quad (6)$$

Such kind of deformation, associated to a displacement field of the form (3), is called an antiplane shear.

The infinitesimal strain tensor is denoted $\boldsymbol{\varepsilon}(\mathbf{u})=(\varepsilon_{ij}(\mathbf{u}))$ and the stress field by $\boldsymbol{\sigma}=(\boldsymbol{\sigma}_{ij})$. We also denote by $\mathbf{E}(\varphi)=(E_i(\varphi))$ the electric field and by $\mathbf{D}=(D_i)$ the electric displacement field. Here and below, in order to simplify the notation, we do not indicate the dependence of various functions on x_1, x_2, x_3 or t and we recall that

$$\varepsilon_{ij}(\mathbf{u})=\frac{1}{2}(u_{i,j}+u_{j,i}),\ \ E_i(\varphi)=-\varphi_{,i}.$$

The material's is modeled by the following electro-elastic constitutive law with Tresca friction law

$$\sigma=\lambda\,(\mathbf{tr}\ \in(\mathbf{u}))\mathbf{I}+2\mu\in(\mathbf{u})-\mathbf{E}^*\mathbf{E}(\varphi),\quad (7)$$

$$\mathbf{D}=\mathbf{E}\boldsymbol{\varepsilon}(\mathbf{u})+\beta\mathbf{E}(\varphi),\quad (8)$$

where λ and μ are the Lame coefficients $\varepsilon(\mathbf{u})=(\in ij\ (\mathbf{u}))$, $\mathbf{I}$ is the unit tensor in IR^3, β is the electric permittivity constant, $\mathbf{E}$ represents the third-order piezoelectric tensor and $\mathbf{E}^*$ is its transpose. In the antiplane context (5), (6), using the constitutive equations (7), (8) it follows that the stress field and the electric displacement field are given by

$$\boldsymbol{\sigma}=\begin{pmatrix}0 & 0 & \boldsymbol{\sigma}_{13}\\ 0 & 0 & \boldsymbol{\sigma}_{23}\\ \boldsymbol{\sigma}_{31} & \boldsymbol{\sigma}_{32} & 0\end{pmatrix},\quad (9)$$

$$\mathbf{D}=\begin{pmatrix}eu,_1-\beta\varphi,_1\\ eu,_2-\beta\varphi,_2\\ 0\end{pmatrix}\quad (10)$$

where

$$\boldsymbol{\sigma}_{13}=\boldsymbol{\sigma}_{31}=\mu\partial_{x_1}u$$

and

$$\boldsymbol{\sigma}_{23}=\boldsymbol{\sigma}_{32}=\mu\partial_{x_2}u.$$

We assume that

$$\mathbf{E}\boldsymbol{\varepsilon}=\begin{pmatrix}e(\varepsilon_{13}+\varepsilon_{31})\\ e(\varepsilon_{23}+\varepsilon_{32})\\ e\varepsilon_{33}\end{pmatrix}\ \forall\boldsymbol{\varepsilon}=(\varepsilon_{ij})\in\mathbf{S}^3,\quad (11)$$

where e is a piezoelectric coefficient. We also assume that the coefficients μ,β and e depend on the spatial variables x_1, x_2, but are independent on the spatial variable x_3. Since $\mathbf{E}\boldsymbol{\varepsilon}\cdot\mathbf{v}=\boldsymbol{\varepsilon}\cdot\mathbf{E}^*\mathbf{v}$ for all $\boldsymbol{\varepsilon}\in\mathbf{S}^3$, $\mathbf{v}\in IR^3$, it follows from (e) that

$$\mathbf{E}^*\mathbf{v}=\begin{pmatrix}0 & 0 & ev_1\\ 0 & 0 & ev_2\\ ev_1 & ev_2 & ev_3\end{pmatrix}\ \forall\mathbf{v}=(v_i)\in\mathrm{IR}^3.\quad (12)$$

We assume that the process is mechanically quasistatic and electrically static and therefore is governed by the equilibrium equations

$$\mathbf{Div}\sigma+\mathbf{f}_0=0,\ D_{i,i}-q_0=0\ \text{ in }\ B\times[0,T]\ ,$$

where $\mathbf{Div}\sigma=(\sigma_{ij,j})$ represents the divergence of the tensor field σ. Taking into account (1), (3), (5), (6), (9) and (10), the equilibrium equations above reduce to the following scalar equations

$$\mathbf{div}(\mu\nabla u+e\nabla\varphi)+f_0=0,\ in\ \Omega\times[0,T],\ (13)$$

$$\mathbf{div}(e\nabla u-\beta\nabla\varphi)=q_0,\ in\ \Omega\times[0,T].\ (14)$$

Here and below we use the notation

$\mathbf{div}\tau = \tau_{1,1} + \tau_{1,2}$ in $\tau = (\tau_1(x_1, x_2), \tau_2(x_1, x_2))$

and

$\nabla v = (v_1, v_2)$, $\partial_{\iota}/v = v_1 v_1 + v_2 v_2$ for

$v = v(x_1, x_2)$

We now describe the boundary conditions. During the process the cylinder is clamped on $\Gamma_1 \times (-\infty, +\infty)$ and the electric potential vanish on $\Gamma_1 \times (-\infty, +\infty)$; thus, (5) and (6) imply that

$$u = 0 \text{ on } \Gamma_1 \times [0, T], \quad (15)$$

$$\varphi = 0 \text{ on } \Gamma_a \times [0, T]. \quad (16)$$

Let $\boldsymbol{\nu}$ denote the unit normal on $\Gamma \times (-\infty, +\infty)$. We have

$$\nu = (\nu_1, \nu_2, 0) \text{ with } \nu_i = \nu_i(x_1, x_2) : \Gamma \to \mathrm{IR}, \ i = 1, 2. \quad (17)$$

For a vector $\mathbf{v}$ we denote by v_ν and $\mathbf{v}_\tau$ its normal and tangential components on the boundary, given by

$$v_\nu = \mathbf{v} \cdot \boldsymbol{\nu}, \ \ \mathbf{v}_\tau = \mathbf{v} - v_\nu \boldsymbol{\nu}. \quad (18)$$

For a given stress field $\boldsymbol{\sigma}$ we denote by σ_ν and $\boldsymbol{\sigma}_\tau$ the normal and the tangential components on the boundary, that is

$$\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}, \ \ \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}. \quad (19)$$

From (9), (10) and (17) we deduce that the Cauchy stress vector and the normal component of the electric displacement field are given by

$$\boldsymbol{\sigma}\boldsymbol{\nu} = (0, 0, \mu\partial_\nu u + e\partial_\nu \varphi), \ \ \mathbf{D} \cdot \boldsymbol{\nu} = e\partial_\nu u - \beta\partial_\nu \varphi. \quad (20)$$

Taking into account (2), (4) and (20), the traction condition on $\Gamma_2 \times (-\infty, \infty)$ and the electric conditions conditions on $\Gamma_b \times (-\infty, \infty)$ are given by

$$\mu\partial_\nu u + e\partial_\nu \varphi = f_2 \text{ on } \Gamma_2 \times 0, T], [(21)$$

$$e\partial_\nu u - \beta\partial_\nu \varphi = q_2 \text{ on } \Gamma_b \times [0, T).(22)$$

We now describe the frictional contact condition and the electric conditions on $\Gamma_3 \times (-\infty, +\infty)$. First, from (5) and (17) we infer that the normal displacement vanishes, $u_\nu = 0$, which shows that the contact is bilateral, that is, the contact is kept during all the process. Using now (5) and (17)-(19) we conclude that

$$\mathbf{u}_\tau = (0, 0, u), \ \ \boldsymbol{\sigma}_\tau = (0, 0, \sigma_\tau) \quad (23)$$

where

$$\sigma_\tau = (0, 0, \mu\partial_\nu u + e\partial_\nu \varphi).$$

We assume that the friction is invariant with respect to the x_3 axis and is modeled with Tresca's friction law, that is

$$\boldsymbol{\sigma}_\tau(t) = \begin{cases} 0, \text{if } u = 0, \\ -g|u|^{s-1}, & \text{if } u \neq 0 \end{cases} \text{ on } \Gamma_3 \times (0, T).(24)$$

Here $g : \Gamma_3 \to \mathrm{IR}_+$ is a given function, the friction bound, and $\mathbf{u}_\tau$ represents the tangential velocity on the contact boundary. Using now (23) it is straightforward to see that the friction law (24) implies

$$\mu\partial_\nu u + e\partial_\nu \varphi = \begin{cases} 0, \text{if } u = 0, \\ -g|u|^{s-1}, & \text{if } u \neq 0 \end{cases} \text{ on } \Gamma_3 \times (0, T).(25)$$

Next, since the foundation is electrically conductive and the contact is bilateral, we assume that the normal component of the electric displacement field or the free charge is proportional to the difference between the potential on the foundation and the body's surface. Thus,

$$\mathbf{D} \cdot \boldsymbol{\nu} = q_2 \text{ on } \Gamma_b \times (0, T),$$

Then, we get

$$\begin{pmatrix} eu_{,1} - \beta\varphi_{,1} \\ eu_{,2} - \beta\varphi_{,2} \\ 0 \end{pmatrix} . \boldsymbol{\nu} = q_2 \text{ on } \Gamma_b \times (0, T). \ (26)$$

Finally, we use (20) and the previous equality to obtain

$$e\partial_\nu u - \beta\partial_\nu \varphi = q_2 \text{ on } \Gamma_b \times (0, T). \quad (27)$$

We collect the above equations and conditions to obtain the following mathematical model which describes the antiplane shear of an electro-viscoelastic cylinder in frictional contact with a conductive foundation.

Problem P. Find the displacement field $u : \Omega \to \mathrm{IR}$ and the electric potential $\varphi : \Omega \to \mathrm{IR}$ such that

$$\mathbf{div}(\mu\nabla u) + \mathbf{div}(e\nabla\varphi) + f_0 = 0, \ in\ \Omega, \ (28)$$

$$\mathbf{div}(e\nabla u) - \mathbf{div}(\alpha\nabla\varphi) = q_0 \ \mathbf{in}\ \Omega, \ (29)$$

$$u = 0 \text{ on } \Gamma_1, (30)$$

$$\mu\partial_\nu u + e\partial_\nu \varphi = f_2 \text{ on } \Gamma_2, (31)$$

$$\mu\partial_\nu u + e\partial_\nu \varphi = \begin{cases} 0, \text{if } u = 0, \\ -g|u|^{s-1}, & \text{if } u \neq 0 \end{cases} \text{ on } \Gamma_3, (25)$$

$$\varphi = 0 \text{ on } \Gamma_a, \ (33)$$

$$e\partial_\nu u - \alpha\partial_\nu \varphi = q_2 \quad \text{on } \Gamma_b. \quad (34)$$

Note that once the displacement field u and the electric potential φ which solve Problem P are known, then the stress tensor σ and the electric displacement field $\mathbf{D}$ can be obtained by using the constitutive laws (9) and (10), respectively.

3. VARIATIONAL FORMULATION

For a real Banach space $(X, \|\cdot\|_X)$ we use the usual notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$ where $1 \le p \le \infty,\ k = 1,2,...$; we also denote by $C([0,T];X)$ the space of continuous and continuously differentiable functions on $[0,T]$ with values in X, with the norm

$$\|x\|_{C([0,T];X)} = \max_{t\in[0,T]} \|x(t)\|_X$$

and we use the standard notations for the Lebesgue space $L^2(0,T;X)$ as well as the Sobolev space $W^{1,2}(0,T;X)$. In particular, recall that the norm on the space $L^2(0,T;X)$ is given by the formula

$$\|u\|^2_{L^2(0,T;X)} = \int_0^T \|u(t)\|_X^2\, dt$$

and the norm on the space $W^2(0,T;X)$ is given by the formula

$$\|u\|^2_{W^{1,2}(0,T;X)} = \int_0^T \|u(t)\|_X^2\, dt + \int_0^T \|\dot u(t)\|_X^2\, dt. \quad (38)$$

Finally, we suppose the argument X when $X = IR$; thus, for example, we use the notation $W^2(0,T)$ for the space $W^2(0,T;IR)$ and the notation $\|\cdot\|_{W^2(0,T)}$ for the norm $\|\cdot\|_{W^2(0,T;\mathbb{R})}$.

In the study of the **Problem P** we assume that the viscosity coefficient satisfy:

and the electric permittivity coefficient satisfy

$$\beta \in L^\infty(\Omega) \text{ and there exists } \beta^* > 0 \text{ such that } \beta(\mathbf{x}) \ge \beta^* \text{ a.e. } \mathbf{x} \in \Omega. \quad (39)$$

We also assume that the Lame coefficient and the piezoelectric coefficient satisfy

$$\mu \in L^\infty(\Omega) \quad (40)$$

and

$$\mu(\mathbf{x}) > 0 \ \text{ a.e. } \mathbf{x} \in \Omega, (41)$$

$$e \in L^\infty(\Omega). (42)$$

The forces, tractions, volume and surface free charge densities have the regularity

$$f_0 \in L^2(\Omega), (43)$$
$$f_2 \in L^2(\Gamma_2), (44)$$
$$q_0 \in L^2(\Omega), (45)$$
$$q_2 \in L^2(\Gamma_b). (46)$$

The friction bound function g satisfies the following properties

$$g \in L^\infty(\Gamma_3) \text{ and } g(\mathbf{x}) \ge 0 \text{ a.e. } \mathbf{x} \in \Gamma_3. (47)$$

and, moreover,

$$a_\mu(u_0, v)_V + j(v) \ge (f(0), v)_V \quad \forall v \in V. (48)$$

We define now the functional $j : V \to \mathbb{R}_+$ given by the formula

$$j(v) = \frac{1}{s+1}\int_{\Gamma_3} g\,|v|^{s+1}\, da \quad \forall v \in V. (49)$$

We also define the mappings $f \in V$ and $q \in W$, respectively, by

$$(f, v)_V = \int_\Omega f_0 v\, dx + \int_{\Gamma_2} f_2 v\, da, (50)$$

and

$$(q, \psi)_W = \int_\Omega q_0 \psi\, dx - \int_{\Gamma_b} q_2 \psi\, da, (51)$$

for all $v \in V$, $\psi \in W$ and $t \in [0,T]$. The definition of f and q are based on Riesz's representation theorem; moreover, it follows from assumptions by (42)-(43), that the integrals above are well-defined and

$$f \in L^2(\Omega), (52)$$
$$q \in L^2(\Omega). (53)$$

Next, we define the bilinear forms a_μ : $V \times V \to \mathbb{R}$, a_e : $V \times W \to \mathbb{R}$, , and a_α : $W \times W \to \mathbb{R}$, by equalities

$$a_\mu(u,v) = \int_\Omega \mu \nabla u \cdot \nabla v\, dx, (54)$$
$$a_e(u,\varphi) = \int_\Omega e \nabla u \cdot \nabla \varphi\, dx, (55)$$
$$a_\alpha(\varphi,\psi) = \int_\Omega \beta \nabla \varphi \cdot \nabla \psi\, dx, (56)$$

for all $u, v \in V$, $\varphi, \psi \in W$. Assumptions (49)-(51) imply that the integrals above are well defined and, using (37) and (18), it follows that the forms a_μ and a_e are continuous; moreover, the forms a_μ and a_α are symmetric and, in addition, the form a_α is W -elliptic, since

$$a_\alpha(\psi,\psi) \ge \alpha^* \|\psi\|_W^2 \quad \forall \psi \in W. \quad (57)$$

4. MAIN RESULTS

The variational formulation of **Problem P** is based on the

Lemma 1 For all (u,φ) in space $X = V \times W$, then we get

$$\int_{\Omega} \mu\nabla u.\nabla(v-u)dx + \int_{\Omega} e\nabla\varphi.\nabla(v-u)dx + \frac{1}{s+1}\int_{\Gamma_3}(|v|^{s+1} - |u|^{s+1})da \geq$$
$$\int_{\Omega} f_0(v-u)dx + \int_{\Gamma_2} f_2(v-u)da, \forall(v,\psi) \in X = V \times W, \forall(u,\varphi) \in X = V \times W.(58)$$

Proof. We introduce relation (50) in the previous relation, then, we have

$$a_{\mu}(u,v-u) + a_e(\varphi,v-u) + j(v) - j(u) \geq (f,v-u)_V, \forall v \in V, \forall\varphi \in W,(71)$$

which conclude the proof of lemma 1.

Lemma 2. For all element $\psi \in W$ and for all $(u,\varphi) \in V \times W$, then, we have

$$a_e(\varphi,\psi) - a_{\beta}(u,\psi) = (q,\psi)_W, \forall\psi \in W.(72)$$

Proof. It is immediately by using (29), (33) and (34).

We collect the above equations and conditions to obtain the following variational formulation which describes the antiplane shear of an electro-viscoelastic cylinder in frictional contact with a conductive foundation.

Problem PV 1**.** Find a displacement field $u:\Omega \to V$ and an electric potential field $\varphi:\Omega \to W$ such that

$$a_{\mu}(u,v-u) + a_e(\varphi,v-u) + j(v) - j(u) \geq (f,v-u)_V, \forall v \in V, \forall\varphi \in W,(77)$$

$$a_e(\varphi,\psi) - a_{\beta}(u,\psi) = (q,\psi)_W, \forall\psi \in W.(78)$$

Let now using the bilinear form:

$$a(.,.): X \times X \to R$$
$$(x,y) \mapsto a(x,y) = a_{\mu}(u,v) + a_e(\varphi,v) + a_{\beta}(\varphi,\psi) - a_e(u,\psi), \forall x = (u,\varphi) \in X, \forall y = (v,\psi) \in X,(79)$$

the functional

$$J(.): X \to R$$
$$x \mapsto J(x) = j(u), \forall x = (u,\varphi) \in X,(80)$$

and the function

$$F = (f,q) \in X.(81)$$

Now, using notations (79)-(81), the Problem (77)-(78) take the final form:

Problem PV 2**.** Find a couple $x = (u,\varphi) \in X$ such that

$$a\ (x,y-x) + J(x) - J(y) \geq (F,y-x)_X, \forall y \in X.(82)$$

Theorem 3. The **Problem** PV 1 and **Problem** PV 2 are equivalent.

Proof. We have two step to proof our Theorem.

Step 1: Problem PV 1 $\Rightarrow$ **Problem** PV 2

In the first step we ill suppose that $x = (u,\varphi) \in X$ is solution of **Problem** PV 1**.** We change in (78) the element $\psi \in W$ by $(\psi - \varphi) \in W$ and we add the resulting equation to the two sides of the inequality (77), hence, we obtain:

$$a_{\mu}(u,v-u) + a_e(\varphi,v-u) + a_{\beta}(\varphi,\psi-\varphi) - a_{\beta}(u,\psi-\varphi) +$$
$$+ j(v) - j(u) \geq (f,v-u)_V + (q,\psi-\varphi)_W, \forall v \in V, \forall\varphi \in W.(83)$$

Using now notations (79), (80) and (81) then for all $\psi \in W$ and for all $y \in X$, we get

$$a\ (x,y-x) + J(x) - J(y) \geq (F,y-x)_X, \forall y \in X.(84)$$

which conclude the proof of the first step.

Step 2: Problem PV 2 $\Rightarrow$ **Problem** PV 1

In ths second step we will suppose that $x = (u,\varphi) \in X$ is solution of **Problem** PV 2**.** We change the bilinear form $a(.,.)$ by (79), $(F,y-x)_X$ by (81) and the functional $J(.)$ by (80); then, for all $(v,\psi) \in X$, we obtain

$$a_{\mu}(u,v-u) + a_e(\varphi,v-u) + a_{\beta}(\varphi,\psi-\varphi) +$$
$$+ j(v) - j(u) \geq (f,v-u)_V + (q,\psi-\varphi)_W, \forall v \in V, \forall\varphi \in W.(85)$$

We test in the last inequality (85) with $\psi = \varphi$, then we obtain (77). Next, we take $v = u$ and $\psi - \varphi = \varphi \pm \psi - \varphi$ in (84), it follows that for all $\psi \in W$:

$$a_{\beta}(\varphi,\pm\psi) - a_e(\varphi,\pm\psi) \geq (q,\pm\psi)_W, \forall v \in V, \forall\varphi \in W,(86)$$

which conclude the proof of the second. Then, the **Problem** PV 1 and **Problem** PV 2 are equivalent.

Our main existence and uniqueness result, which we state now and prove in the next section, is the following:

Theorem 4. Assume that (39)-(57) hold. Then the variational **Problem** PV 2 possesses a unique solution $x = (u,\varphi) \in X$ satisfies

$$a\ (x,y-x) + J(x) - J(y) \geq (F,y-x)_X, \forall y \in X.(87)$$

We note that an element $x = (u,\varphi)$ which solves Problem PV 1 is scalled a weak solution of the antiplane contact Problem PV 1. We conclude by Theorem 3 that the element $x = (u,\varphi)$ also solves Problem PV 2, then the element x is called a weak solution of the antiplane contact

Problem $PV\ 2$. Hence, the antiplane contact Problem P has a unique weak solution, provided that (39)-(57).

Proof of Theorem 4.

The Proof of Theorem 4 which will be carried out in several steps and it is immediately to obtain our result of existence and uniqueness of the weak solution.

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