Second Order Impulsive Functional Differential Equations with Variable Times and State-Dependent Delay

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***Abstract*—In this paper, we establish sufficient conditions for the existence of solutions for a class of initial value problem for impulsive functional differential equations with variable times involving infinite state-dependent delay.**

***Index Terms*—Differential equation, state-dependent delay, fixed point, infinite delay, impulses, variable times.**

I. INTRODUCTION

THIS paper deals with the existence of solutions to the initial value problems (IVP for short) for the second differential equations with variable times and state dependent delay of the form,

$$y''(t) = f(t, y_{\rho(t,y_t)}), \text{ a.e. } t \in J = [0, b], \tag{1}$$

$$t \neq \tau_k(y(t)),\ k = 1, \dots, m, y(t^+) = I_k(y(t)), \tag{2}$$

$$t = \tau_k(y(t)),\ k = 1, \dots, m, \tag{3}$$

$$y'(t^+) = \overline{I}_k(y(t)),\ t = \tau_k(y(t)),\ k = 1, \dots, m, \tag{4}$$

$$y(t) = \phi(t), \quad t \in (-\infty, 0], \tag{5}$$

$$y'(0) = \eta, \tag{6}$$

where $f : J \times \mathcal{B} \to \mathbb{R}$, $\rho : J \times \mathcal{B} \to (-\infty, b]$, $I_k, \overline{I}_k : \mathbb{R} \to \mathbb{R},\ k = 1, \dots, m$ are given continuous functions, $\phi \in \mathcal{B}$, $y(t^+) = \lim_{h\to 0^+} y(t+h)$ and $y(t^-) = \lim_{h\to 0^-} y(t+h)$ represent the right and left hand limits of $y(t)$ at t and $\mathcal{B}$ is a phase space to be specified later. For any function y and any $t \in J$, we denote by y_t the element of $\mathcal{B}$ defined by $y_t(\theta) = y(t+\theta)$ for $\theta \in (-\infty, 0]$. We assume that the histories y_t belong to $\mathcal{B}$.

The notion of the phase space $\mathcal{B}$ plays an important role in the study of both qualitative and quantitative theory. A usual choice is a semi-normed space satisfying suitable axioms, which was introduced by Hale and Kato [15] (see also Kappel and Schappacher [20] and Schumacher [30]. For a detailed discussion on this topic we refer the reader to the book by Hino *et al*. [18]. For the case where the impulses are absent, an extensive theory has been developed for the problem (1)-(6). We refer to Hale and Kato [15], Corduneanu and Lakshmikantham [9], Hino *et al*. [18], Lakshmikantham *et al* [25].

Impulsive differential equations have become more important in recent years in some mathematical models of real processes and phenomena studied in control, physics, chemistry, population dynamics, biotechnology and economics. There has been a significant development in impulse theory, in recent years, especially in the area of impulsive differential equations with fixed moments, see the monographs of Benchohra *et al.* [5], Lakshmikantham *et al.* [25] and Samoilenko and Perestyuk [29] and the references therein. The theory of impulsive differential equations with variable times is relatively less developed due to the difficulties created by the state-dependent impulses. Some interesting results have been done by Bajo and Liz [1], Benchohra *et al.* [3], [7] and Benchohra and Ouahab [8], Frigon and O'Regan [10], [11], [12], Graef and Ouahab [13], Kaul *et al.* [21], Kaul and Liu [22], [23], Lakshmikantham *et al.* [26], [27] and Liu and Ballinger [28]. The results of the present paper extend those considered in the above cite literature for constant delay. Our approach here is based on the nonlinear alternative of Leray-Schauder type [14].

II. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. By $C(J,\mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\| := \sup\{|y(t)| : t \in J\}.$$

In this paper, we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato in [15] and follow the terminology used in [19], but we will add some transformations. Thus $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ will be a seminormed linear space of functions mapping $(-\infty,0]$ into $\mathbb{R}$.

$L^1(J,\mathbb{R})$ denotes the Banach space of measurable functions $y : J \longrightarrow \mathbb{R}$ which are Lesbegue integrable normed by

$$\|y\|_{L^1} = \int_0^b |y(t)|dt.$$

$AC^1(J,\mathbb{R})$ denote the space for all differentiable functions whose first derivative is absolutely continuous.

Definition II.1. *The map $f : J \times \mathcal{B} \to \mathbb{R}$ is said to be Carathéodory if:*

(i) *The function $t \longmapsto f(t,u)$ is measurable for each $u \in \mathcal{B}$*
(ii) *The function $u \longmapsto f(t,u)$ is continuous for a.e. $t \in J$.*

Consider the sets

$$PC = \Big\{y : [0,b] \to \mathbb{R} : y \text{ which there exist } 0 < t_1 < t_2 < ... < t_{m+1} = b \text{ such that } t_k = \tau_k(y(t_k^-)) \text{ and } y(t_k^+), y(t_k^-) \text{ exists with, } y(t_k^-) = y(t_k)\ k = 1,\dots,m,\ y_k \in C(J_k,\mathbb{R})\Big\},$$

where y_k is the restriction of y to $J_k = (t_k, t_{k+1}]$, $k = 1,\dots,m$,
and

$$B_b = \{y : (-\infty,b] : y|_{(-\infty,0]} \in \mathcal{B} \text{ and } y|_J \in PC\}.$$

Let $\|.\|_b$ the seminorm in B_b defined by

$$\|y\|_b = \|y_0\|_{\mathcal{B}} + \sup\{|y(t)| : 0 \le t \le b\},\ y \in B_b.$$

For the definition of the phase space $\mathcal{B}$ we introduce the following axioms.

(A_1) If $y : (-\infty,b) \to \mathbb{R}, b > 0,\ y_0 \in \mathcal{B}$, the following conditions hold :
(i) $y_t \in \mathcal{B}$,
(ii) There exists a positive constant H such that $|y(t)| \le H\|y_t\|_{\mathcal{B}}$,
(iii) There exist two functions $K(\cdot), M(\cdot) : J \to \mathbb{R}^+$, independent of y, with K continuous and M locally bounded such that :

$$\|y_t\|_{\mathcal{B}} \le K(t)\sup\{\ |y(s)| : 0 \le s \le t\} + M(t)\|y_0\|_{\mathcal{B}}.$$

(A_2) The space $\mathcal{B}$ is complete.

Denote $K_b = \sup\{K(t) : t \in J\}$ and $M_b = \sup\{M(t) : t \in J\}$.

$\mathcal{B} = \{y : (-\infty,0] \to \mathbb{R}, y$ is continuous every where except for a finite number of points $\bar{t}$ at which $y(\bar{t}^+), y(\bar{t}^-)$ exist and $y(\bar{t}^-) = y(\bar{t})\}$

Definition II.2. *A function $y \in B_b \bigcap \bigcup_{i=1}^m AC^1((t_i,t_{i+1}),\mathbb{R})$ is to be a solution of (1)-(6) if y satisfies $y''(t) = f(t, y_{\rho(t,y_t)})$ a.e $t \in J = [0,b]$, $t \neq \tau_k(y(t))$, $k = 1,\dots,m$, the conditions $y(t^+) = I_k(y(t))$, $y'(t^+) = \overline{I}_k(y(t))$ $t = \tau_k(y(t))$, $k = 1,\dots,m$, and $y(t) = \phi(t)$, $t \in (-\infty,0]$, $y'(0) = \eta$.*

We are now in a position to state and prove our result for the problem $(1)-(6)$.

III. Existence of Solutions

In this section we will present an existence result for the problem (1)-(6). First, we introduce the following hypotheses.

(H_ϕ) The function $t \to \phi_t$ is continuous from $\mathcal{R}(\rho^-) = \{\rho(s,\varphi) : (s,\varphi) \in J \times \mathcal{B}, \rho(s,\varphi) \le 0\}$ into $\mathcal{B}$ and there exists a continuous and bounded function $L^\phi : \mathcal{R}(\rho^-) \to (0,\infty)$ such that $\|\phi_t\|_{\mathcal{B}} \le L^\phi(t)\|\phi\|_{\mathcal{B}}$ for every $t \in \mathcal{R}(\rho^-)$.
(H_1) The function $f : J \times \mathcal{B} \to \mathbb{R}$ is Carathéodory,
(H_2) There exists $p \in L^1(J,\mathbb{R}_+)$ and $\psi : [0,\infty) \to (0,\infty)$ continuous and nondecreasing such that

$$|f(t,u)| \le p(t)\psi(\|u\|_{\mathcal{B}}) \text{ for each } t \in J \text{ and all } u \in \mathcal{B},$$

with

$$K_b\int_0^b p(s)ds < \int_C^\infty \frac{dx}{\psi(x)},$$

where $C = M_b\|\phi\|_B + K_b|\phi(0)|$.

(H_3) The functions $\tau_k \in C^1(\mathbb{R}, \mathbb{R})$ for $k = 1, \ldots, m$. Moreover

$$0 < \tau_1(x) < \ldots < \tau_m(x) < b \text{ for all } x \in \mathbb{R}.$$

(H_4) For all $x \in \mathbb{R}$

$$\tau_k(I_k(x)) \leq \tau_k(x) < \tau_{k+1}(I_k(x)) \text{ for } k = 1, \ldots, m.$$

(H_5) For all $a \in J$ fixed, $y \in B_b$ and for a.e. $t \in J$ we have

$$\tau_k'(y(t)) \int_a^t (t-s) f(s, y_{\rho(t,y_t)}) ds \neq 1 \text{ for } k = 1, \ldots, m.$$

(H_6) The functions $I_k,\ \overline{I}_k,\ k = 1, 2, \ldots, m$ are continuous.

The next result is consequence of the phase space axioms.

Lemma III.1. *If $y : (-\infty, b] \to \mathbb{R}$ is a function such that $y_0 = \phi$ and $y|_J \in PC(J, \mathbb{R})$, then*

$$\|y_s\|_{\mathcal{B}} \leq (M_b + L^{\phi})\|\phi\|_{\mathcal{B}} + K_b \sup\{\|y(\theta)\|;\, \theta \in [0, \max\{0, s\}]\}, \quad s \in \mathcal{R}(\rho^-) \cup J,$$

where

$$L^{\phi} = \sup_{t \in \mathcal{R}(\rho^-)} L^{\phi}(t).$$

Remark III.1. *We remark that condition (H_ϕ) is satisfied by functions which are continuous and bounded. In fact, if the space $\mathcal{B}$ satisfies axiom C_2 in [19] then there exists a constant $L > 0$ such that $\|\phi\|_{\mathcal{B}} \leq L \sup\{\|\phi(\theta)\| : \theta \in [-\infty, 0]\}$ for every $\phi \in \mathcal{B}$ that is continuous and bounded (see [19] Proposition 7.1.1) for details. Consequently,*

$$\|\phi_t\|_{\mathcal{B}} \leq L \frac{\sup_{\theta \leq 0} \|\phi(\theta)\|}{\|\phi\|_{\mathcal{B}}} \|\phi\|_{\mathcal{B}}, \text{ for every } \phi \in \mathcal{B}\backslash\{0\}.$$

Theorem III.1. *Assume that hypotheses (H_ϕ), (H_1)-(H_6). Then the problem (1)-(6) has at least one solution on $(-\infty, b]$.*

Proof. The proof will be given in a couple of steps.
Step 1: Consider the initial value problem

$$y''(t) = f(t, y_{\rho(t,y_t)}), \quad a.e.\ t \in J, \tag{7}$$
$$y(t) = \phi(t), \quad t \in (-\infty, 0], \tag{8}$$
$$y'(0) = \eta. \tag{9}$$

Set

$$\tilde{C} = \{y : (-\infty, b] : y|_{(-\infty,0]} \in \mathcal{B} \text{ and } y \in C(J, \mathbb{R})\}.$$

Define the operator $N : \tilde{C} \to \tilde{C}$ by:

$$N(y)(t) = \begin{cases} \phi(t), \\ \quad \text{if} \quad t \in (-\infty, 0], \\ \phi(0) + t\eta + \displaystyle\int_0^t (t-s) f(s, y_{\rho(s,y_s)}) ds \\ \quad \text{if} \quad t \in [0, b]. \end{cases}$$

Clearly the fixed point of N are solutions to (7)–(9). Let $x(.) : (-\infty, b] \to \mathbb{R}$ be the function defined by:

$$x(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty, 0], \\ \phi(0) + t\eta, & \text{if } t \in [0, b]. \end{cases}$$

Then $x_0 = \phi$. For each $z \in \mathcal{B}_b$ with $z_0 = 0$, we denote by $\overline{z}$ the function defined by

$$\overline{z}(t) = \begin{cases} 0, & \text{if } t \in (-\infty, 0], \\ z(t), & \text{if } t \in [0, b]. \end{cases}$$

If $y(\cdot)$ satisfies the integral equation

$$y(t) = \phi(0) + t\eta + \int_0^t (t-s) f(s, y_{\rho(s,y_s)}) ds.$$

We can decompose $y(.)$ into $y(t) = \overline{z}(t) + x(t)$, $0 \leq t \leq b$, which implies $y_t = \overline{z}_t + x_t$, for every $t \in [0, b]$, and the function $z(\cdot)$ satisfies

$$z(t) = \int_0^t (t-s) f(s, \overline{z}_{\rho(s,\overline{z}_s + x_s)} + x_{\rho(s,\overline{z}_s+x_s)}) ds.$$

Let

$$\mathcal{C}_0 = \{z \in \tilde{C} : z_0 = 0\}.$$

Let $\|.\|_0$ be the norm in $\mathcal{C}_0$ defined by

$$\begin{aligned} \|z\|_0 &= \|z_0\|_{\mathcal{B}} + \sup\{|z(s)| : 0 \leq s \leq b\} \\ &= \sup\{|z(s)| : 0 \leq s \leq b\} = \|z\|_b. \end{aligned}$$

We define the operator $P : \mathcal{C}_0 \to \mathcal{C}_0$ by

$$P(z)(t) = \int_0^t (t-s) f(s, \overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)}) ds.$$

Obviously the operator N has a fixed point is equivalent to P has one, so we need to prove that P has a fixed point. We shall show that P satisfies the assumptions of Leray-Schauder alternative. The proof will be given in several Claims.

Claim 1: *P is continuous.*

Let $\{y^n\}$ be a sequence such that $y_0^n = 0$ and $y^n \to y$ in $\mathcal{C}_0$. Then for each $t \in J$,

$$\begin{aligned}|(Py^n)(t) - (Py)(t)| &\leq \int_0^t |(t-s)f(s, y^n_{\rho(s,y^n(s)}) \\ &\quad - f(s, y_{\rho(s,y_s)})|ds \\ &\leq \int_0^b |t-s||f(s, y^n_{\rho(s,y_q(s)}) \\ &\quad - f(s, y_{\rho(s,y_s)})|.\end{aligned}$$

Since f is Carathéodory we have $f(s, y^n_{\rho(s,y_q(s))}) \to f(s, y_{\rho(s,y_s)})$ as $n \to \infty$, for every $s \in J$. Now a standard application of the Lebesgue dominated convergence theorem implies that

$$\|Py^n - Py\|_0 \to 0 \quad \text{as} \quad n \to \infty.$$

and then P is continuous.

Claim 2: *P maps bounded set into a bounded set of $\mathcal{C}_0$.*

Indeed it is enough to show that for any $q > 0$, there exists a positive constant ℓ such that for each $z \in B_q = \{z \in \mathcal{C}_0 : \|z\|_0 \leq q\}$, one has $\|P(z)\|_0 \leq \ell$.
Let $z \in B_q$ by (H_2) we have for each $t \in J$,

$$\begin{aligned}|P(z)(t)| &\leq \int_0^t |t-s|\|f(s, \overline{z}_{\rho(s,\overline{z}_s+x_s)} \\ &\quad + x_{\rho(s,\overline{z}_s+x_s)})\|ds \\ &\leq \int_0^t |t-s|p(s)\psi(\|\overline{z}_{\rho(s,\overline{z}_s+x_s)} \\ &\quad + x_{\rho(s,\overline{z}_s+x_s)}\|)ds \\ &\leq \psi(K_b q \\ &\quad + K_b|\phi(0)| + M_b\|\phi\|_{\mathcal{B}})) \int_0^t |t-s|p(s)ds \\ &\leq \psi(K_b q + K_b|\phi(0)| \\ &\quad + M_b\|\phi\|_{\mathcal{B}})) \int_0^b |t-s|p(s)ds \\ &= l.\end{aligned}$$

Claim 3: *P maps bounded sets into equicontinuous sets of $\mathcal{C}_0$.*

Let $\tau_1, \tau_2 \in [0, b]$, $\tau_1 < \tau_2$, let B_q a bounded set of $\mathcal{C}_0$ as in Claim 2, and let $z \in B_q$, then

$$\begin{aligned}&|(Pz)(\tau_2) - (Pz)(\tau_1)| \leq |\int_0^{\tau_2} (\tau_2 - s)f(s, y_{\rho(s,y_s)})ds \\ &- \int_0^{\tau_1} (\tau_2 - s)f(s, y_{\rho(s,y_s)})ds| \\ &\leq |\int_0^{\tau_2} (\tau_2 - s)f(s, y_{\rho(s,y_s)})ds \\ &- \int_0^{\tau_2} (\tau_1 - s)f(s, y_{\rho(s,y_s)})ds| \\ &+ |\int_0^{\tau_2} (\tau_1 - s)f(s, y_{\rho(s,y_s)})ds \\ &- \int_0^{\tau_1} (\tau_1 - s)f(s, y_{\rho(s,y_s)})ds| \\ &\leq |\int_0^{\tau_2} (\tau_2 - \tau_1)f(s, y_{\rho(s,y_s)})ds| \\ &+ |\int_{\tau_1}^{\tau_2} (\tau_1 - s)f(s, y_{\rho(s,y_s)})ds| \\ &\leq \psi(\|y_{\rho(s,y_s)}\|)(|\int_0^{\tau_2} p(s)(\tau_2 - \tau_1)ds| \\ &+ |\int_{\tau_1}^{\tau_2} p(s)(\tau_1 - s)ds|) \\ &\leq \psi(q^*)(|\int_0^{\tau_2} p(s)(\tau_2 - \tau_1)ds| \\ &+ |\int_{\tau_1}^{\tau_2} p(s)(\tau_1 - s)ds|).\end{aligned}$$

Where

$$q^* = K_b q + K_b|\phi(0)| + M_b\|\phi\|_{\mathcal{B}}$$

We see that $|(Pz)(\tau_2) - (Pz)(\tau_1)|$ tend to zero independently of $z \in B_q$ as $\tau_2 \to \tau_1$. As a consequence of claims 1 to 3 together with the Ascoli-Arzela theorem we can conclude that P is continuous and completely continuous.

Claim 4: *A priori bounds.*

Now it remains to show that the set

$$\mathcal{E} = \{z \in \mathcal{C}_0 : z = \lambda P(z) \text{ for some } 0 < \lambda < 1\}$$

is bounded. Let $z \in \mathcal{E}$, then $z = \lambda P(z)$ for some $0 < \lambda < 1$. Thus, for each $t \in J$,

$$z(t) = \lambda \int_0^t f(s, \overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)})ds.$$

Then for each $t \in J$, we have

$$|z(t)| \leq \lambda \int_0^t p(s)\psi(\|\overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)}\|_{\mathcal{B}})ds, \tag{10}$$

but

$$\|\overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)}\|_{\mathcal{B}} \leq \|\overline{z}_{\rho(s,\overline{z}_s+x_s)}\|_{\mathcal{B}}$$

$$+\|x_{\rho(s,\overline{z}_s+x_s)}\|_{\mathcal{B}}$$

$$\leq K(t)\sup\{|z(s)|:0\leq s\leq t\}$$
$$+M(t)\|z_0\|_{\mathcal{B}}+K(t)\sup\{|x(s)|$$
$$:0\leq s\leq t\}+M(t)\|x_0\|_{\mathcal{B}}$$

$$\leq K_b\sup\{|z(s)|:0\leq s\leq t\}$$
$$+M_b\|\phi\|_{\mathcal{B}}+K_b|\phi(0)|,$$

and then

$$\|\overline{z}_{\rho(s,\overline{z}_s+x_s)}+x_{\rho(s,\overline{z}_s+x_s)}\|_{\mathcal{B}}\leq K_b\sup\{|z(s)| \quad (11)$$
$$:0\leq s\leq t\}+M_b\|\phi\|_{\mathcal{B}}+K_b|\phi(0)|. \quad (12)$$

If we name $w(t)$ the right hand side of (11), then we have

$$\||\overline{z}_{\rho(s,\overline{z}_s+x_s)}+x_{\rho(s,\overline{z}_s+x_s)}\|_{\mathcal{B}}\leq w(t).$$

Therefore (10) becomes

$$|z(t)|\leq\int_0^t p(s)\psi(w(s))ds. \quad (13)$$

Using (13) in the definition of w, we have

$$w(t)\leq K_b\int_0^t p(s)\psi(w(s))ds+M_b\|\phi\|_{\mathcal{B}}+K_b|\phi(0)|.$$

Let us take the right hand-side of the last inequality as $v(t)$. Then we have

$$w(t)\leq v(t)\quad\text{for all }\ t\in J,$$

$$v(0)=K_b|\phi(0)|+M_b\|\phi\|_{\mathcal{B}},$$

and

$$v'(t)=K_bp(t)\psi(w(t)),\quad a.e.\ \ t\in J.$$

Using the nondecreasing character of ψ we get

$$v'(t)\leq K_bp(t)\psi(v(t)),\quad a.e.\ \ t\in J.$$

That is

$$\frac{v'(t)}{\psi(v(t))}\leq K_bp\left(t\right),\quad a.e.\ \ t\in J.$$

Integrating from 0 to t we get

$$\int_0^t\frac{v'(s)}{\psi(v(s))}ds\leq K_b\int_0^t p\left(s\right)ds.$$

By a change of variable and using $(\mathcal{H}_2)$ we get

$$\int_{v(0)}^{v(t)}\frac{du}{\psi(u)}\leq K_b\int_0^b p\left(s\right)ds<\int_C^{\infty}\frac{du}{\psi(u)}.$$

Hence there exists a constant K_* such that

$$v(t)\leq K_*\quad\text{for all }\ t\in J.$$

and hence $\|\overline{z}_t+x_t\|_{\mathcal{C}}\leq w(t)\leq K_*,\ t\in J$. From (13) we have that

$$\|z\|_0\leq\psi(K_*)\int_0^b p(s)ds:=K_1.$$

Set

$$U=\{y\in\mathcal{C}_0:\sup\{|z(t)|,\ 0\leq t\leq b\}\leq K_1+1.$$

From the choice of U, there is no $y\in\partial U$ such that $y=\lambda P(y)$ for some $\lambda\in[0,1]$. The nonlinear alternative of Leray-Schauder type [14] implies that P has a fixed point, hence N has a fixed point which is a solution of problem (7)-(9). Denote this solution by y_1.

Define the function

$$r_{k,1}(t)=\tau_k(y_1(t))-t\ \text{for}\ \ t\geq 0.$$

(H_3) implies that

$$r_{k,1}(0)\neq 0\ \text{for}\ \ k=1,\ldots,m.$$

If

$$r_{k,1}(t)\neq 0\ \text{on}\ J\ \text{for}\ k=1,\ldots,m,$$

i.e

$$t\neq\tau_k(y_1(t))\ \text{on}\ J\ \text{for}\ k=1,\ldots,m,$$

then y_1 is solution of the problem (1)-(6).

Now we consider the case when $r_{1,1}(t)=0$ for some $t\in J$. Since $r_{1,1}(0)\neq 0$ and $r_{1,1}$ is continuous, there exists $t_1>0$ such that

$$r_{1,1}(t_1)=0\ \text{and}\ r_{1,1}(t)\neq 0\ \text{for all}\ t\in[0,t_1).$$

Thus by (H_3) we have

$$r_{k,1}(t)\neq 0\ \text{for all}\ t\in[0,t_1)\ \text{and}\ k=1,\ldots,m.$$

Step 2: Consider the following problem

$$y''(t)=f(t,y_{\rho(t,y_t)}),\ \text{for a.e.,}\ t\in[t_1,b] \quad (14)$$

$$y(t_1^+)=I_1(y_1(t_1^-)) \quad (15)$$

$$y'(t_1^+)=\overline{I}_1(y_1(t_1^-)), \quad (16)$$

$$y(t)=y_*(t),\ t\in(-\infty,t_1]. \quad (17)$$

Where

$$y_*(t)=\begin{cases}y_1(t), & \text{if}\ \ t\in[0,t_1]\\ \phi(t), & \text{if}\ \ t\in(-\infty,0]\end{cases}$$

Let

$$\mathcal{C}_1 = \{y \in \mathcal{C}((t_1, b], \mathbb{R}), y(t_1^+) \ \text{ exist}\},$$

and

$$\mathcal{C}_* = \{y : (-\infty, b] \to \mathbb{R} : y \in C((-\infty, t_1], \mathbb{R}) \cap \mathcal{C}_1\}.$$

Consider the operator $N_1 : \mathcal{C}_* \to \mathcal{C}_*$ defined by:

$$N(y)(t) = \begin{cases} y_*(t), \\ \text{ if } \quad t \in (-\infty, t_1], \\ I_1(y_1(t_1)) + (t - t_1)\left(\overline{I}_1(y_1(t_1))\right) \\ \text{ if } \ t \in [t_1, b] \\ + \displaystyle\int_{t_1}^{t} (t-s) f(s, y_{\rho(s,y_s)}) ds. \end{cases}$$

Let $x(.) : (-\infty, b] \to \mathbb{R}$ be the function defined by

$$x(t) = \begin{cases} I_1 \qquad (y_1(t_1)) + (t - t_1)\left(\overline{I}_1(y_1(t_1))\right), \\ \qquad \text{ if } \ t \in (t_1, b], \\ y_*(t), \\ \qquad \text{ if } \ t \in (-\infty, t_1]. \end{cases}$$

Then $x_{t_1} = y_1$. For each $z \in \mathcal{C}_*$ with $z_{t_1} = 0$, we denote by $\overline{z}$ the function defined by

$$\overline{z}(t) = \begin{cases} 0, & \text{if } \ t \in (-\infty, t_1], \\ z(t), & \text{if } \ t \in [t_1, b]. \end{cases}$$

If $y(\cdot)$ satisfies the integral equation

$$y(t) = I_1(y_1(t_1)) + (t - t_1)\left(\overline{I}_1(y_1(t_1))\right) + \int_{t_1}^{t} (t-s) f(s, y_{\rho(s,y_s)}) ds.$$

We can decompose $y(.)$ into $y(t) = \overline{z}(t) + x(t)$, $t_1 \le t \le b$, which implies $y_t = \overline{z}_t + x_t$, for every $t \in [t_1, b]$, and the function $z(\cdot)$ satisfies

$$z(t) = \int_{t_1}^{t} (t-s) f(s, \overline{z}_{\rho(s,\overline{z}_s + x_s)} + x_{\rho(s,\overline{z}_s + x_s)}) ds.$$

Let

$$\mathcal{C}_{t_1} = \{z \in \mathcal{C}_*, z(t_1) = 0\}.$$

Let the operator $P : \mathcal{C}_{t_1} \to \mathcal{C}_{t_1}$ by

$$P_1(z)(t) = \int_{t_1}^{t} (t-s) f(s, \overline{z}_{\rho(s,\overline{z}_s + x_s)} + x_{\rho(s,\overline{z}_s + x_s)}) ds.$$

As in Step 1 we can show that P_1 is continuous and completely continuous and if z is a solution for the equation $z = \lambda P_1(z)$, for some $\lambda \in (0, 1)$ there exist $K_{1_*} > 0$ such that

$$\|z\|_\infty \le K_{1_*} > 0.$$

Set

$$U_1 = \{y \in \mathcal{C}_{t_1} : \sup\{\|z(t)\| : t_1 \le t \le b\} \le K_{1_*} + 1.$$

As a consequence of Leray-Schauder's nonlinear alternative type we deduce that P has a fixed point z in U_1. Thus N_1 has a fixed point y which is a solution of problem (14)-(17), denote this solution by y_2.

Define

$$r_{k,2}(t) = \tau_k(y_2(t)) - t \text{ for } \ t \ge t_1.$$

If

$$r_{k,2}(t) \ne 0 \text{ on } (t_1, b] \text{ for } k = 1, \ldots, m,$$

then

$$y(t) = \begin{cases} y_1(t), & \text{if } t \in [0, t_1], \\ y_2(t), & \text{if } t \in (t_1, b], \end{cases}$$

is solution of the problem (1)-(6). It remains to consider the case when

$$r_{2,2}(t) = 0, \text{ for some } t \in (t_1, b].$$

By (H_4) we have

$$\begin{array}{rcl} r_{2,2}(t_1^+) & = & \tau_2(y_2(t_1^+)) - t_1 \\ & = & \tau_2(I_1(y_1(t_1^-))) - t_1 \\ & > & \tau_1(y_1(t_1^-)) - t_1 \\ & = & r_{1,1}(t_1) = 0. \end{array}$$

Since $r_{2,2}$ is continuous, there exists $t_2 > t_1$ such that $r_{2,2}(t_2) = 0$ and $r_{2,2}(t) \ne 0$ for all $t \in (t_1, t_2)$. By (H_3) we have :

$$r_{k,2} \ne 0 \text{ for all } t \in (t_1, t_2) \text{ and } k = 2, \ldots, m.$$

Suppose now that there exists $\overline{s} \in (t_1, t_2]$ such that $r_{1,1}(\overline{s}) = 0$. From (H_4) it follows that

$$\begin{array}{rcl} r_{1,2}(t_1^+) & = & \tau_1(y_2(t_1^+)) - t_1 \\ & = & \tau_1(I_1(y_1(t_1^-))) - t_1 \\ & \le & \tau_1(y_1(t_1)) - t_1 \\ & = & r_{1,1}(t_1) = 0. \end{array}$$

Thus the function $r_{1,2}$ attains a nonnegative maximum at some point $s_1 \in (t_1, b]$. Since

$$y_2'(t) = \int_{t_1}^{t} f(s, y_{2_{\rho(s,y_{2_s})}}) ds$$

then

$$r_{1,2}'(s_1) = \tau_1'(y_2(s_1) y_2'(s_1) - 1 = 0.$$

Therefore

$$\tau_1'(y_2(s_1)) \int_{t_1}^{s_1} f(s, y_{2_{\rho(s,y_{2_s})}}) ds = 1,$$

which contradicts (H_5)

Step 3: We continue this process and taking into account that $y_{m+1} := y|_{[t_m,b]}$ is a solution to the problem

$$y''(t) = f(t, y_{\rho(t,y_t)}), \text{ for a.e., } t \in (t_m, b], \quad (18)$$

$$y(t_m^+) = I_m(y_m(t_m^-)), \quad (19)$$

$$y'(t_m^+) = \overline{I}_m(y_m(t_m^-)). \quad (20)$$

The solution y of the problem (1)-(6) is then defined by

$$y(t) = \begin{cases} y_1(t), & \text{if } t \in (-\infty, t_1], \\ y_2(t), & \text{if } t \in (t_1, b], \\ \dots & \\ y_{m+1}(t), & \text{if } t \in (t_m, b]. \end{cases}$$

IV. Example

To apply our results, we consider the functional differential equation with variables times and state-dependent delay of the form :

$$y''(t) = \frac{(y(t - \sigma(y(t))))^2}{(t^2+1)(t+2)(1 + (y(t - \sigma(y(t))))^2)} a.e, \quad (21)$$

$$t \in [0,1], t \neq \tau_k(y(t)), k = 1, \dots, m, \quad (22)$$

$$\Delta y|_{t=\tau_k(y(t))} = I_k(y(t)), t = \tau_k(y(t)), k = 1, \dots, m, \quad (23)$$

$$\Delta y'|_{t=\tau_k(y(t))} = \overline{I}_k(y(t)), t = \tau_k(y(t)), k = 1, \dots, m, \quad (24)$$

$$y(t) = \phi(t), t \in (-\infty, 0], \quad (25)$$

$$y'(0) = \eta, \quad (26)$$

where $\sigma \in C(\mathbb{R}, [0,\infty))$. Set $\gamma > 0$. For the phase space, we choose $\mathcal{B}$ to be defined by

$$\mathcal{B}_\gamma = \{y \in PC((-\infty, 0], \mathbb{R}) : \lim_{\theta \to -\infty} e^{\gamma\theta} y(\theta) \text{ exists}\}$$

with the norm

$$\|y\|_\gamma = \sup_{\theta \in (-\infty, 0]} e^{\gamma\theta} |y(\theta)|,$$

where

$$PC((-\infty,0],\mathbb{R}) = \Big\{ y : (-\infty, 0] \to \mathbb{R} : y \text{ is continuous at } t \neq \tilde{t}_k,\ y(\tilde{t}_k^-) = y(\tilde{t}_k) \text{ and } y(\tilde{t}_k^+) \text{ exists for all } k = 1, \dots, m \Big\}.$$

Let $y : (-\infty, b] \to \mathbb{R}$ be such that $y_0 \in \mathcal{B}_\gamma$. Then

$$\begin{aligned} \lim_{\theta \to -\infty} e^{\gamma\theta} y(\theta) &= \lim_{\theta \to -\infty} e^{\gamma\theta} y(t+\theta) \\ &= \lim_{\theta \to -\infty} e^{\gamma(\theta - t)} y(\theta) \\ &= e^{\gamma\ t} \lim_{\theta \to -\infty} e^{\gamma\theta} y_0(\theta). \end{aligned}$$

Hence $y_t \in \mathcal{B}_\gamma$. Finally we prove that

$$\|y_t\|_\gamma \leq K(t) \sup\{\ |y(s)| : 0 \leq s \leq t\} + M(t)\|y_0\|_\gamma,$$

where $K = M = H = 1$. We have $y(t) = y(t + \phi)$. If $t + \theta \leq 0$ we get

$$\|y_t(\theta)\| \leq \sup\{\ |y(s)| : -\infty \leq s \leq 0\}.$$

For $t + \theta \geq 0$ we have

$$\|y_t(\theta)\| \leq \sup\{\ |y(s)| : 0 \leq s \leq t\}.$$

Thus for all $t + \theta \in [0,1]$, we get

$$\|y_t(\theta)\| \leq \sup\{\ |y(s)| : -\infty \leq s \leq 0\} + \sup\{\ |y(s)| : 0 \leq s \leq t\}.$$

Thus

$$\|y_t\|_\gamma \leq \|y\|_0 + \sup\{\ |y(s)| : 0 \leq s \leq t\}.$$

It is clear that $(\mathcal{B}_\gamma, \|y\|_\gamma)$ is a Banach space. We can conclude that $\mathcal{B}_\gamma$ is a phase space.

Set

$$f(t,u) = \frac{(u(0))^2}{(t^2+1)(t+2)(1+(u(0)^2)}, \quad (t,u) \in [0,1] \times \mathcal{B}_\gamma,$$

$$\rho(t,u) = t - \sigma(u(0)), \quad (t,u) \in [0,1] \times \mathcal{B},$$

$$\tau_k(x) = 2k - \frac{1}{2^{k+1}(1+x^2)},$$

$$I_k(x) = d_k x,$$

$$\overline{I_k}(x) = \overline{d_k} x.$$

From the the definition of τ_k we have $\tau_k(x) \neq 0$ and

$$\tau_{k+1}(x) - \tau_k(x) = 2 + \frac{1}{2^{k+2}(1+x^2)} > 0 \text{ for all } x \in \mathbb{R} \text{ and } k = 1, \dots, m.$$

So

$$0 < \tau_1(x) < \tau_2(x) < \tau_3(x) < \ldots < \tau_k(x) \text{ for all } x \in \mathbb{R}.$$

Also

$$\tau_k(I_k(x)) - \tau_k(x) = \frac{(b_k^2 - 1)x^2}{2^{k+1}(1+x^2)(1+b_k^2 x^2)} \leq 0$$

and

$$\tau_{k+1}(I_k(x)) - \tau_k(x) = \frac{2^{k+3}(1+x^2)(1+b_k^2x^2)+1+(2b_k^2-1)x^2}{2^{k+1}(1+x^2)(1+b_k^2x^2)} > 0$$

for all $x \in \mathbb{R}$ and $k = 1, \dots, m$. Thus

$$\tau_k(I_k(x)) \leq \tau_k(x) \leq \tau_{k+1}(I_k(x)) \text{ for all } x \in \mathbb{R} \text{ and } k = 1, \dots, m.$$

We can easily show that

$$\begin{aligned}|\tau_k'(x)\int_a^t f(s, y_{\rho(s,y_s)})ds| &\leq |\tau_k'(x) \\ &\cdot |\int_a^t |f(s, y_{\rho(s,y_s)})|ds \\ &\leq \frac{1}{2}|t-a| \\ &< 1.\end{aligned}$$

Assume that $p(t) = \frac{1}{t^2+1}$ and $\psi(x) = 1$.Then

$$|f(t,u)| \leq \frac{1}{t^2+1}\psi(\|u\|_b) \text{ for all } (t,u) \in [0,1] \times \mathcal{B}_b$$

$$\int_c^{+\infty} \frac{du}{\psi(u)} = +\infty$$

It is clear that all conditions of Theorem III.1 are satisfied. Hence problem (20)-(26) has at least one solution defined on $]-\infty, b]$.

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